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PLANE AND SPHERICAL

TRIGONOMETRY.

BY

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BOSTON, U.S.A.:
PUBLISHED BY GINN & COMPANY.
1892.

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PREFACE.

IN preparing this work the aim has been to furnish just so much of Trigonometry as is actually taught in our best schools and colleges. Consequently, development of functions in series and all other investigations that are important only for the special student have been omitted. The principles have been unfolded with the utmost brevity consistent with simplicity and clearness, and interesting problems have been selected with a view to awaken a real love for the study. Much time and labor have been spent in devising the simplest proofs for the propositions, and in exhibiting the best methods of arranging the logarithmic work.

The object of the work on Surveying and Navigation is to present these subjects in a clear and intelligible way, according to the best methods in actual use; and also to present them in so small a compass that students in general may find the time to acquire a competent knowledge of these very interesting and important studies.

The author is under particular obligation for assistance to G. A. Hill, A.M., of Cambridge, Mass., and to Prof. James L. Patterson, of Lawrenceville, N.J.

G. A. WENTWORTH.

PHILLIPS EXETER ACADEMY,
September, 1882.

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PLANE TRIGONOMETRY.

CHAPTER I.

TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES.

§ 1. DEFINITIONS.

THE sides and angles of a plane triangle are so related that any three given parts, provided at least one of them is a side, determine the shape and the size of the triangle.

Geometry shows how, from three such parts, to *construct* the triangle and find the values of the unknown parts.

Trigonometry shows how to *compute* the unknown parts of a triangle from the numerical values of the given parts.

Geometry shows in a general way that the sides and angles of a triangle are mutually dependent. Trigonometry begins by showing the exact nature of this dependence in the *right triangle*, and for this purpose employs the *ratios of its sides.*

Let MAN (Fig. 1) be an acute angle. If from any points B, D, F, ... in one of its sides perpendiculars BC, DE, FG, ... are let fall to the other side, then the right triangles ABC, ADE, AFG, ... thus formed have the angle A common, and are therefore mutually equiangular and similar. Hence, the ratios of their corresponding sides, pair by pair, are equal. That is,

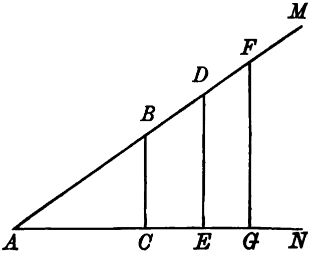

Fig. 1.

$$\frac{AC}{AB}=\frac{AE}{AD}=\frac{AG}{AF}; \quad \frac{AC}{BC}=\frac{AE}{DE}=\frac{AG}{FG}; \text{ etc.}$$

Hence, for every value of an acute angle A there are certain *numbers* that express the values of the ratios of the sides *in all right triangles that have this acute angle A.*

There are altogether six different ratios:

I. The ratio of the opposite leg to the hypotenuse is called the *Sine* of A, and is written $\sin A$.

II. The ratio of the adjacent leg to the hypotenuse is called the *Cosine* of A, and written $\cos A$.

III. The ratio of the opposite leg to the adjacent leg is called the *Tangent* of A, and written $\tan A$.

IV. The ratio of the adjacent leg to the opposite leg is called the *Cotangent* of A, and written $\cot A$.

V. The ratio of the hypotenuse to the adjacent leg is called the *Secant* of A, and written $\sec A$.

VI. The ratio of the hypotenuse to the opposite leg is called the *Cosecant* of A, and written $\csc A$.

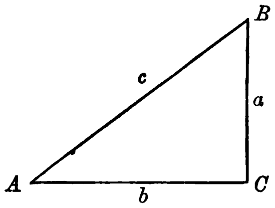

Fig. 2.

In the right triangle ABC (Fig. 2) let a, b, c denote the lengths of the sides opposite to the acute angles A, B, and the right angle C, respectively, these lengths being all expressed in terms of a common unit. Then,

$$\sin A = \frac{a}{c} = \frac{\text{opposite leg}}{\text{hypotenuse}}. \qquad \cos A = \frac{b}{c} = \frac{\text{adjacent leg}}{\text{hypotenuse}}.$$

$$\tan A = \frac{a}{b} = \frac{\text{opposite leg}}{\text{adjacent leg}}. \qquad \cot A = \frac{b}{a} = \frac{\text{adjacent leg}}{\text{opposite leg}}.$$

$$\sec A = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent leg}}. \qquad \csc A = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite leg}}.$$

These six ratios are called the **Trigonometric Functions** of the angle A.

EXERCISE I.

1. What are the functions of the other acute angle B of the triangle ABC (Fig. 2)?

2. Prove that if two angles, A and B, are complements of each other (*i.e.*, if $A + B = 90°$), then,

$$\sin A = \cos B, \quad \tan A = \cot B, \quad \sec A = \csc B;$$
$$\cos A = \sin B, \quad \cot A = \tan B, \quad \csc A = \sec B.$$

3. Find the values of the functions of A, if a, b, c respectively have the following values:

(i.) 3, 4, 5. (iii.) 8, 15, 17. (v.) 3.9, 8, 8.9.
(ii.) 5, 12, 13. (iv.) 9, 40, 41. (vi.) 1.19, 1.20, 1.69.

4. What condition must be fulfilled by the lengths of the three lines a, b, c (Fig. 2) in order to make them the sides of a right triangle? Is this condition fulfilled in Example 3?

5. Find the values of the functions of A, if a, b, c respectively have the following values:

(i.) $2mn$, $m^2 - n^2$, $m^2 + n^2$. (iii.) pqr, qrs, rsp.
(ii.) $\frac{2xy}{x-y}$, $x+y$, $\frac{x^2+y^2}{x-y}$. (iv.) $\frac{mn}{pq}$, $\frac{mv}{sq}$, $\frac{nr}{ps}$.

6. Prove that the values of a, b, c, in (i.) and (ii.), Example 5, satisfy the condition necessary to make them the sides of a right triangle.

7. What equations of condition must be satisfied by the values of a, b, c, in (iii.) and (iv.), Example 5, in order that the values may represent the sides of a right triangle?

Compute the functions of A and B when,

8. $a = 24$, $b = 143$.
9. $a = 0.264$, $c = 0.265$.
10. $b = 9.5$, $c = 19.3$.
11. $a = \sqrt{p^2 + q^2}$, $b = \sqrt{2pq}$.
12. $a = \sqrt{p^2 + pq}$, $c = p + q$.
13. $b = 2\sqrt{pq}$, $c = p + q$.

- Compute the functions of A when,

14. $a = 2b$.

15. $a = \frac{2}{3}c$.

16. $a + b = \frac{5}{4}c$.

17. $a - b = \frac{c}{4}$.

18. Find a if $\sin A = \frac{3}{5}$ and $c = 20.5$.

19. Find b if $\cos A = 0.44$ and $c = 3.5$.

20. Find a if $\tan A = \frac{11}{8}$ and $b = 2\frac{5}{11}$.

21. Find b if $\cot A = 4$ and $a = 17$.

22. Find c if $\sec A = 2$ and $b = 20$.

23. Find c if $\csc A = 6.45$ and $a = 35.6$.

Construct a right triangle: given,

24. $c = 6$, $\tan A = \frac{3}{2}$.

25. $a = 3.5$, $\cos A = \frac{1}{2}$.

26. $b = 2$, $\sin A = 0.6$.

27. $b = 4$, $\csc A = 4$.

28. In a right triangle, $c = 2.5$ miles, $\sin A = 0.6$, $\cos A = 0.8$; compute the legs.

29. Construct (with a protractor) the $\angle$s 20°, 40°, and 70°; determine their functions by measuring the necessary lines, and compare the values obtained in this way with the more correct values given in the following table:

	sin	*cos*	*tan*	*cot*	*sec*	*csc*
20°	0.342	0.940	0.364	2.747	1.064	2.924
40°	0.643	0.766	0.839	1.192	1.305	1.556
70°	0.940	0.342	2.747	0.364	2.924	1.064

30. Find, by means of the above table, the legs of a right triangle if $A = 20°$, $c = 1$; also, if $A = 20°$, $c = 4$.

31. In a right triangle, given $a = 3$ and $c = 5$; find the hypotenuse of a similar triangle in which $a = 240{,}000$ miles.

32. By dividing the length of a vertical rod by the length of its horizontal shadow, the tangent of the angle of elevation of the sun at the time of observation was found to be 0.82. How high is a tower, if the length of its horizontal shadow at the same time is 174.3 yards?

§ 2. Representation of Functions by Lines.

The functions of an angle, being ratios, are numbers; but we may represent them by *lines* if we first choose a unit of length, and then construct right triangles, such that the denominators of the ratios shall be equal to this unit. The most convenient way to do this is as follows:

About a point O (Fig. 3) as a centre, with a radius equal to one unit of length, describe a circle and draw two diameters AA' and BB' perpendicular to each other.

The circle with radius equal to 1 is called a *unit* circle, AA' the *horizontal*, and BB' the *vertical* diameter.

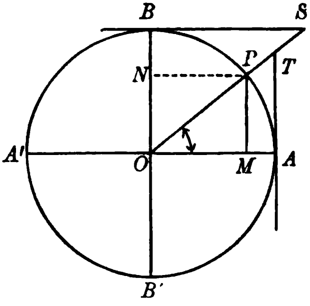

Fig. 3.

Let AOP be an acute angle, and let its value (in degrees, etc.) be denoted by x. We may regard the $\angle\, x$ as generated by a radius OP that revolves about O from the position OA to the position shown in the figure; viewed in this way, OP is called the *moving* radius.

Draw $PM \perp$ to OA. In the rt. $\triangle\, OPM$ the hypotenuse $OP = 1$; therefore, $\sin x = PM$; $\cos x = OM$.

Since PM is equal to ON, and ON is the projection of OP on BB', and since OM is the projection of OP on AA', therefore, *in a unit circle,*

$\sin x =$ projection of moving radius on vertical diameter;
$\cos x =$ projection of moving radius on horizontal diameter.

Through A and B draw tangents to the circle meeting OP, produced in T and S, respectively; then, in the rt. $\triangle\, OAT$, the leg $OA = 1$, and in the rt. $\triangle\, OBS$, the leg $OB = 1$; while the $\angle\, OSB = \angle\, x$ (why?). Therefore,

$$\tan x = AT; \quad \sec x = OT; \quad \cot x = BS; \quad \csc x = OS.$$

These six *line* values (as they may be termed) of the functions are all expressed in terms of the radius of the circle as a unit; and it is clear that as the angle varies in value the line values of the functions will always remain equal *numerically* to the ratio values. Hence, in studying the changes in the functions as the angle is supposed to vary, we may employ the simpler line values instead of the ratio values.

EXERCISE II.

1. Represent by lines the functions of a larger angle than that shown in Fig. 3.

2. Show that $\sin x$ is less than $\tan x$.

3. Show that $\sec x$ is greater than $\tan x$.

4. Show that $\csc x$ is greater than $\cot x$.

Construct the angle x if,

5. $\tan x = 3$. 7. $\cos x = \frac{1}{2}$. 9. $\sin x = 2\cos x$.

6. $\csc x = 2$. 8. $\sin x = \cos x$. 10. $4\sin x = \tan x$.

11. Show that the sine of an angle is equal to one-half the chord of twice the angle.

12. Find x if $\sin x$ is equal to one-half the side of a regular inscribed decagon.

13. Given x and y ($x + y$ being less than 90°); construct the value of $\sin(x+y) - \sin x$.

14. Given x and y ($x + y$ being less than 90°); construct the value of $\tan(x+y) - \sin(x+y) + \tan x - \sin x$.

Given an angle x; construct an angle y such that,

15. $\sin y = 2\sin x$. 17. $\tan y = 3\tan x$.

16. $\cos y = \frac{1}{2}\cos x$. 18. $\sec y = \csc x$.

19. Show by construction that $2\sin A > \sin 2A$.

20. Given two angles A and B ($A + B$ being less than 90°); show that $\sin(A+B) < \sin A + \sin B$.

21. Given $\sin x$ in a unit circle; find the length of a line corresponding in position to $\sin x$ in a circle whose radius is r?

22. In a right triangle, given the hypotenuse c, and also $\sin A = m$, $\cos A = n$; find the legs.

§ 3. CHANGES IN THE FUNCTIONS AS THE ANGLE CHANGES.

If we suppose the $\angle AOP$, or x (Fig. 4) to increase gradually by the revolution of the moving radius OP about O, the point P will move along the arc AB towards B, T will move along the tangent AT away from A, S will move along the tangent BS towards B, and M will move along the radius OA towards O.

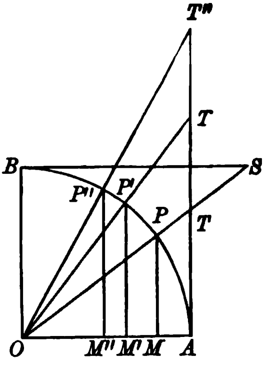

Fig. 4.

Hence, the lines PM, AT, OT will gradually increase in length, and the lines OM, BS, OS, will gradually decrease. That is,

As an acute angle increases, its sine, tangent, and secant also increase, while its cosine, cotangent, and cosecant decrease.

On the other hand, if we suppose x to decrease gradually, the reverse changes in its functions will occur.

If we suppose x to decrease to 0°, OP will coincide with OA and be parallel to BS. Therefore, PM and AT will vanish, OM will become equal to OA, while BS and OS will each be infinitely long, and be represented in value by the symbol ∞.

And if we suppose x to increase to 90°, OP will coincide with OB and be parallel to AT. Therefore, PM and OS will each be equal to OB, OM and BS will vanish, while AT and OT will each be infinite in length.

Hence, as the angle x increases from 0° to 90°,

$\sin x$ increases from 0 to 1,
$\cos x$ decreases from 1 to 0,
$\tan x$ increases from 0 to ∞,
$\cot x$ decreases from ∞ to 0,
$\sec x$ increases from 1 to ∞,
$\csc x$ decreases from ∞ to 1.

The values of the functions of 0° and of 90° are the *limiting* values of the functions of an acute angle. It is evident that (disregarding the limiting values),

Sines and cosines are always less than 1;
Secants and cosecants are always greater than 1;
Tangents and cotangents have all values between 0 and ∞.

REMARK. We are now able to understand why the sine, cosine, etc., of an angle are called *functions* of the angle. By a *function* of any magnitude is meant another magnitude which remains the same so long as the first magnitude remains the same, but changes in value for every change in the value of the first magnitude. This, as we now see, is the relation in which the sine, cosine, etc., of an angle stand to the angle.

§ 4. FUNCTIONS OF COMPLEMENTARY ANGLES.

The general form of two complementary angles is A and $90° - A$.

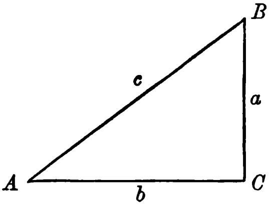

Fig. 5.

In the rt. △ ABC (Fig. 5) $A + B = 90°$, hence $B = 90° - A$. Therefore (§ 1),

$$\begin{aligned}
\sin A &= \cos B = \cos(90° - A),\\
\cos A &= \sin B = \sin(90° - A),\\
\tan A &= \cot B = \cot(90° - A),\\
\cot A &= \tan B = \tan(90° - A),\\
\sec A &= \csc B = \csc(90° - A),\\
\csc A &= \sec B = \sec(90° - A).
\end{aligned}$$

Therefore,

Each function of an acute angle is equal to the co-named function of the complementary angle.

NOTE. *Co*sine, *co*tangent, and *co*secant are sometimes called *co-functions;* the words are simply abbreviated forms of *complement's sine, complement's tangent,* and *complement's secant.*

Hence, also,

Any function of an angle between 45° *and* 90° *may be found by taking the co-named function of the complementary angle between* 0° *and* 45°.

EXERCISE III.

1. Express the following functions as functions of the complementary angle:

$\sin 30^\circ$. $\tan 89^\circ$. $\csc 18^\circ\, 10'$. $\cot 82^\circ\, 19'$.
$\cos 45^\circ$. $\cot 15^\circ$. $\cos 37^\circ\, 24'$. $\csc 54^\circ\, 46'$.

2. Express the following functions as functions of an angle less than 45°:

$\sin 60^\circ$. $\tan 57^\circ$. $\csc 69^\circ\, 2'$. $\cot 89^\circ\, 59'$.
$\cos 75^\circ$. $\cot 84^\circ$. $\cos 85^\circ\, 39'$. $\csc 45^\circ\, 1'$.

3. Given $\tan 30^\circ = \frac{1}{3}\sqrt{3}$; find $\cot 60^\circ$.
4. Given $\tan A = \cot A$; find A.
5. Given $\cos A = \sin 2A$; find A.
6. Given $\sin A = \cos 2A$; find A.
7. Given $\cos A = \sin(45^\circ - \frac{1}{2}A)$; find A.
8. Given $\cot \frac{1}{2}A = \tan A$; find A.
9. Given $\tan(45^\circ + A) = \cot A$; find A.
10. Find A if $\sin A = \cos 4A$.
11. Find A if $\cot A = \tan 8A$.
12. Find A if $\cot A = \tan nA$.

§ 5. RELATIONS OF THE FUNCTIONS OF AN ANGLE.

Since (Fig. 5) $a^2 + b^2 = c^2$, therefore,

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1 \quad \text{or} \quad \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

Therefore (§ 1), $(\sin A)^2 + (\cos A)^2 = 1$;

or, as usually written for convenience,

$$\sin^2 \mathbf{A} + \cos^2 \mathbf{A} = 1. \qquad [1]$$

That is: *The sum of the squares of the sine and the cosine of an angle is equal to unity.*

Formula [1] enables us to find the cosine of an angle when the sine is known, and *vice versa*. The values of $\sin A$ and of $\cos A$ deduced from [1] are:

$$\sin A = \sqrt{1 - \cos^2 A}, \quad \cos A = \sqrt{1 - \sin^2 A}.$$

Since $$\frac{a}{c} \div \frac{b}{c} = \frac{a}{c} \times \frac{c}{b} = \frac{a}{b},$$

therefore (§ 1), $$\tan \mathbf{A} = \frac{\sin \mathbf{A}}{\cos \mathbf{A}}. \qquad [2]$$

That is: *The tangent of an angle is equal to the sine divided by the cosine.*

Formula [2] enables us to find the tangent of an angle when the sine and the cosine are known.

Since $\frac{a}{c} \times \frac{c}{a} = 1$, $\frac{b}{c} \times \frac{c}{b} = 1$, and $\frac{a}{b} \times \frac{b}{a} = 1$,

therefore (§ 1), $$\left.\begin{array}{l} \sin \mathbf{A} \times \csc \mathbf{A} = 1 \\ \cos \mathbf{A} \times \sec \mathbf{A} = 1 \\ \tan \mathbf{A} \times \cot \mathbf{A} = 1 \end{array}\right\}. \qquad [3]$$

That is: *The sine and the cosecant of an angle, the cosine and secant, and the tangent and cotangent, pair by pair, are reciprocals.*

The equations in [3] enable us to find an unknown function contained in any pair of these reciprocals when the other function in this pair is known.

EXERCISE IV.

1. Prove Formulas [1]--[3], using for the functions the line values in unit circle given in § 3.

2. Prove that $1 + \tan^2 A = \sec^2 A$.

   Hint. Divide the terms of the equation $a^2 + b^2 = c^2$ by b^2.

3. Prove that $1 + \cot^2 A = \csc^2 A$.

4. Prove that $\cot A = \frac{\cos A}{\sin A}$.

§ 6. APPLICATION OF FORMULAS [1]–[3].

Formulas [1], [2], and [3] enable us, when any one function of an angle is given, to find all the others. A given value of any one function, therefore, determines all the others.

EXAMPLE 1. Given $\sin A = \frac{2}{3}$; find the other functions.

By [1], $\cos A = \sqrt{1 - \frac{4}{9}} = \sqrt{\frac{5}{9}} = \frac{1}{3}\sqrt{5}$.

By [2], $\tan A = \frac{2}{3} \div \frac{1}{3}\sqrt{5} = \frac{2}{3} \times \frac{3}{\sqrt{5}} = \frac{2}{\sqrt{5}}$.

By [3], $\cot A = \frac{\sqrt{5}}{2}$, $\sec A = \frac{3}{\sqrt{5}}$, $\csc A = \frac{3}{2}$.

EXAMPLE 2. Given $\tan A = 3$; find the other functions.

By [2], $\frac{\sin A}{\cos A} = 3$.

And by [1], $\sin^2 A + \cos^2 A = 1$.

If we solve these equations (regarding $\sin A$ and $\cos A$ as two unknown quantities), we find that,

$$\sin A = 3\sqrt{\tfrac{1}{10}}, \quad \cos A = \sqrt{\tfrac{1}{10}}.$$

Then by [3], $\cot A = \frac{1}{3}$, $\sec A = \sqrt{10}$, $\csc A = \frac{1}{3}\sqrt{10}$.

EXAMPLE 3. Given $\sec A = m$; find the other functions.

By [3], $\cos A = \frac{1}{m}$.

By [1], $\sin A = \sqrt{1 - \frac{1}{m^2}} = \sqrt{\frac{m^2 - 1}{m^2}} = \frac{1}{m}\sqrt{m^2 - 1}$.

By [2], [3], $\tan A = \sqrt{m^2 - 1}$, $\cot A = \frac{1}{\sqrt{m^2 - 1}}$,

$$\csc A = \frac{m}{\sqrt{m^2 - 1}}.$$

EXERCISE V.

Find the values of the other functions when,

1. $\sin A = \frac{12}{13}$.
2. $\sin A = 0.8$.
3. $\cos A = \frac{60}{61}$.
4. $\cos A = 0.28$.
5. $\tan A = \frac{4}{3}$.
6. $\cot A = 1$.
7. $\cot A = 0.5$.
8. $\sec A = 2$.
9. $\csc A = \sqrt{2}$.
10. $\sin A = m$.
11. $\sin A = \frac{2m}{1+m^2}$.
12. $\cos A = \frac{2mn}{m^2+n^2}$.

13. Given $\tan 45° = 1$; find the other functions of 45°.

14. Given $\sin 30° = \frac{1}{2}$; find the other functions of 30°.

15. Given $\csc 60° = \frac{2}{3}\sqrt{3}$; find the other functions of 60°.

16. Given $\tan 15° = 2 - \sqrt{3}$; find the other functions of 15°.

17. Given $\cot 22° 30' = \sqrt{2} + 1$; find the other functions of 22° 30′.

18. Given $\sin 0° = 0$; find the other functions of 0°.

19. Given $\sin 90° = 1$; find the other functions of 90°.

20. Given $\tan 90° = \infty$; find the other functions of 90°.

21. Express the values of all the other functions in terms of $\sin A$.

22. Express the values of all the other functions in terms of $\cos A$.

23. Express the values of all the other functions in terms of $\tan A$.

24. Express the values of all the other functions in terms of $\cot A$.

25. Given $2 \sin A = \cos A$; find $\sin A$ and $\cos A$.

26. Given $4 \sin A = \tan A$; find $\sin A$ and $\tan A$.

27. If $\sin A : \cos A = 9 : 40$, find $\sin A$ and $\cos A$,

28. Transform the quantity $\tan^2 A + \cot^2 A - \sin^2 A - \cos^2 A$ into a form containing only $\cos A$.

29. Prove that $\sin A + \cos A = (1 + \tan A) \cos A$.

30. Prove that $\tan A + \cot A = \sec A \times \csc A$.

§ 7. FUNCTIONS OF 45°.

Let ABC (Fig. 6) be an isosceles right triangle, in which the length of the hypotenuse AB is equal to 1; then AC is equal to BC, and the angle A is equal to 45°. Since $\overline{AC}^2 + \overline{BC}^2 = 1$, therefore $2\,\overline{AC}^2 = 1$, and $AC = \sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2}$.

Therefore (§ 1),

$$\sin 45° = \cos 45° = \tfrac{1}{2}\sqrt{2}.$$
$$\tan 45° = \cot 45° = 1.$$
$$\sec 45° = \csc 45° = \sqrt{2}.$$

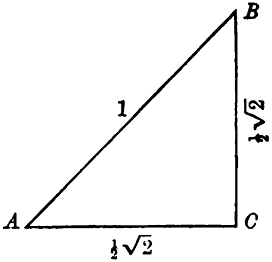

Fig. 6.

§ 8. FUNCTIONS OF 30° AND 60°.

Let ABC be an equilateral triangle in which the length of each side is equal to 1; and let CD bisect the angle C. Then CD is perpendicular to AB and bisects AB; hence, $AD = \frac{1}{2}$, and $CD = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$.

In the right triangle ADC, the angle $ACD = 30°$, and the angle $CAD = 60°$.

Whence (§ 1),

$$\sin 30° = \cos 60° = \tfrac{1}{2}.$$
$$\cos 30° = \sin 60° = \tfrac{1}{2}\sqrt{3}.$$
$$\tan 30° = \cot 60° = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3}.$$
$$\cot 30° = \tan 60° = \sqrt{3}.$$
$$\sec 30° = \csc 60° = \frac{2}{\sqrt{3}} = \tfrac{2}{3}\sqrt{3}.$$
$$\csc 30° = \sec 60° = 2.$$

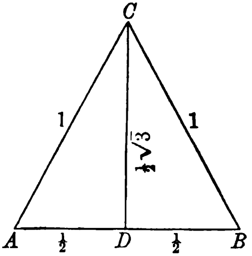

Fig. 7.

The results for sine and cosine of 30°, 45°, and 60° may be easily remembered by arranging them in the following form:

Angle . . .	30°	45°	60°	$\frac{1}{2} = 0.5$
Sine	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2} = 0.70711$
Cosine . . .	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3} = 0.86603$

CHAPTER II.

THE RIGHT TRIANGLE.

§ 9. The Given Parts.

In order to solve a right triangle, two parts besides the right angle must be given, one of them at least being a side.

The two given parts may be :

I. An acute angle and the hypotenuse.
II. An acute angle and the opposite leg.
III. An acute angle and the adjacent leg.
IV. The hypotenuse and a leg.
V. The two legs.

§ 10. Case I.

Given $A = 34° \, 28'$ and $c = 18.75$; required B, a, b.

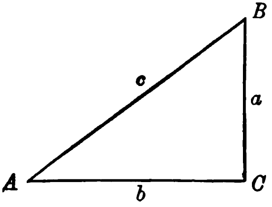

Fig. 8.

1. $B = 90° - A = 55° \, 32'$.

2. $\frac{a}{c} = \sin A$; $\therefore a = c \sin A$.

3. $\frac{b}{c} = \cos A$; $\therefore b = c \cos A$.

$$\begin{array}{ll|ll} \log a & = \log c + \log \sin A & \log b & = \log c + \log \cos A \\ \log c & = \;\; 1.27300 & \log c & = \;\; 1.27300 \\ \log \sin A & = \;\; \underline{9.75276}^{*} & \log \cos A & = \;\; \underline{9.91617} \\ \log a & = \;\; 1.02576 & \log b & = \;\; 1.18917 \\ a & = 10.61 & b & = 15.459 \end{array}$$

---

* For Logarithms, and directions how to use them, see Wentworth and Hill's *Five-place Tables.*

When –10 belongs to a logarithm or cologarithm, and is not written, it must be *remembered* that the logarithm or cologarithm is 10 too large.

§ 11. Case II.

Given $A = 62° 10'$, $a = 78$; find B, b, c.

1. $B = 90° - A = 27° 50'$.
2. $\frac{b}{a} = \cot A$; $\therefore b = a \cot A$.
3. $\frac{a}{c} = \sin A$.

$\therefore\ a = c \sin A$, and $c = \frac{a}{\sin A}$.

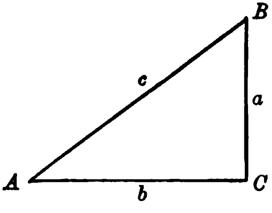

Fig. 9.

$\log b$	$= \log a + \log \cot A$	$\log c$	$= \log a + \operatorname{colog} \sin A$
$\log a$	$= 1.89209$	$\log a$	$= 1.89209$
$\log \cot A$	$= 9.72262$	$\operatorname{colog} \sin A$	$= 0.05340$
$\log b$	$= 1.61471$	$\log c$	$= 1.94549$
b	$= 41.182$	c	$= 88.204$

§ 12. Case III.

Given $A = 50° 2'$, $b = 88$; find B, a, c.

1. $B = 90° - A = 39° 58'$.
2. $\frac{a}{b} = \tan A$; $\therefore a = b \tan A$.
3. $\frac{b}{c} = \cos A$.

$\therefore\ b = c \cos A$, and $c = \frac{b}{\cos A}$.

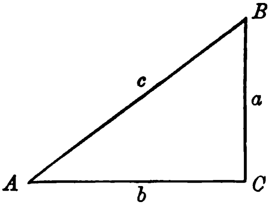

Fig. 10.

$\log a$	$= \log b + \log \tan A$	$\log c$	$= \log b + \operatorname{colog} \cos A$
$\log b$	$= 1.94448$	$\log b$	$= 1.94448$
$\log \tan A$	$= 10.0767$	$\operatorname{colog} \cos A$	$= 0.19223$
$\log a$	$= 2.02118$	$\log c$	$= 2.13671$
a	$= 105.0$	c	$= 137.0$

§ 13. Case IV.

Given $c = 58.40$, $a = 47.55$; find A, B, b.

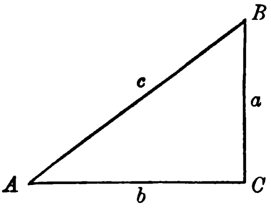

Fig. 11.

1. $\sin A = \frac{a}{c}$.

2. $B = 90° - A$.

3. $\frac{b}{a} = \cot A$; $\therefore b = a \cot A$.

$$\begin{aligned}
\log \sin A &= \log a + \operatorname{colog} c \\
\log a &= 1.67715 \\
\operatorname{colog} c &= \underline{8.23359} \\
\log \sin A &= 9.91074 \\
A &= 54° \, 31' \\
B &= 35° \, 29'
\end{aligned}$$

$$\begin{aligned}
\log b &= \log a + \log \cot A \\
\log a &= 1.67715 \\
\log \cot A &= \underline{9.85300} \\
\log b &= 1.53015 \\
b &= 33.896
\end{aligned}$$

§ 14. Case V.

Given $a = 40$, $b = 27$; find A, B, c.

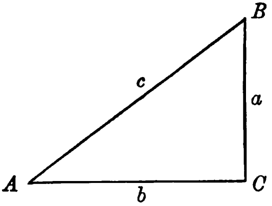

Fig. 12.

1. $\tan A = \frac{a}{b}$.

2. $B = 90° - A$.

3. $\frac{a}{c} = \sin A$.

$\therefore a = c \sin A$; $\therefore c = \frac{a}{\sin A}$.

$$\begin{aligned}
\log \tan A &= \log a + \operatorname{colog} b \\
\log a &= 1.60206 \\
\operatorname{colog} b &= \underline{8.56864} \\
\log \tan A &= 10.17070 \\
A &= 55° \, 59' \\
B &= 34° \, 1'
\end{aligned}$$

$$\begin{aligned}
\log c &= \log a + \operatorname{colog} \sin A \\
\log a &= 1.60206 \\
\operatorname{colog} \sin A &= \underline{0.08152} \\
\log c &= 1.68358 \\
c &= 48.259
\end{aligned}$$

§ 15. General Method of Solving the Right Triangle.

From these five cases it appears that the general method of finding an unknown part in a right triangle is as follows:

Choose from the equation $A + B = 90°$, *and the equations that define the functions of the angles, an equation in which the required part only is unknown; solve this equation, if necessary, to find the value of the unknown part; then compute the value, using logarithms whenever convenient.*

Note 1. In Cases IV. and V. the unknown side may also be found by Geometry, from the equation $a^2 + b^2 = c^2$; whence we obtain

(for Case IV.) $b = \sqrt{c^2 - a^2} = \sqrt{(c+a)(c-a)}$;

(for Case V.) $c = \sqrt{a^2 + b^2}$.

These equations express the values of b and c directly in terms of the two given sides; and if the values of the sides are simple numbers (*e.g.* 5, 12, 13), it is often easier to find b or c in this way. But this value of c is not adapted to logarithms, and this value of b is not so readily worked out by logarithms as the value of b given in § 13.

Note 2. In Case IV. if the given sides (here a and c) are nearly alike in value, then A is near 90°, and its value cannot be accurately found from the tables, because the sines of large angles differ little in value (as is evident from Fig. 4). In this case it is better to find B first, by means of a formula proved later (see page 47); viz.,

$$\tan \tfrac{1}{2} B = \sqrt{\frac{c-a}{c+a}};$$

and to find b by the method given in Note 1, since the same logarithms are used in both cases.

Example. Given $a = 49$, $c = 50$; find A, B, b.

$$\begin{aligned}
\log b &= \tfrac{1}{2}[\log(c-a) + \log(c+a)]\\
c - a &= 1\\
c + a &= 99\\
\log(c-a) &= 0.00000\\
\log(c+a) &= \underline{1.99564}\\
&\ 2\overline{)1.99564}\\
\log b &= 0.99782\\
b &= 9.95
\end{aligned}$$

$$\begin{aligned}
\log\tan \tfrac{1}{2}B &= \tfrac{1}{2}[\log(c-a)\\
&\qquad + \operatorname{colog}(c+a)]\\
\log(c-a) &= 0.00000\\
\operatorname{colog}(c+a) &= \underline{8.00436}\\
&\ 2\overline{)8.00436}\\
\log\tan \tfrac{1}{2}B &= 9.00218\\
\tfrac{1}{2}B &= 5°\ 44'\ 21''\\
B &= 11°\ 29'\\
A &= 78°\ 31'
\end{aligned}$$

§ 16. Area of the Right Triangle.

It is shown in Geometry that the area of a triangle is equal to one-half the product of the base by the altitude.

Therefore, if a and b denote the legs of a right triangle, and F the area,

$$F = \tfrac{1}{2}ab.$$

By means of this formula the area may always be found when a and b are given or have been computed.

For example: Find the area, having given:

Case I. (§ 10).

$A = 34° 28'$, $c = 18.75$.

First find (as in § 10) $\log a$ and $\log b$.

$$\log(F) = \log a + \log b + \operatorname{colog} 2$$

$$\begin{aligned}
\log a &= 1.02578\\
\log b &= 1.18915\\
\operatorname{colog} 2 &= \underline{9.69897}\\
\log(F) &= 1.91390\\
F &= 82.016
\end{aligned}$$

Case IV. (§ 13).

$a = 47.54$, $c = 58.40$.

First find (as in § 13) $\log a$ and $\log b$.

$$\log(F) = \log a + \log b + \operatorname{colog} 2$$

$$\begin{aligned}
\log a &= 1.67715\\
\log b &= 1.53025\\
\operatorname{colog} 2 &= \underline{9.69897}\\
\log(F) &= 2.90637\\
F &= 806.06
\end{aligned}$$

Exercise VI.

1. In Case II. give another way of finding c, after b has been found.
2. In Case III. give another way of finding c, after a has been found.
3. In Case IV. give another way of finding b, after the angles have been found.
4. In Case V. give another way of finding c, after the angles have been found.
5. Given B and c; find A, a, b.
6. Given B and b; find A, a, c.
7. Given B and a; find A, b, c.
8. Given b and c; find A, B, a.

Solve the following triangles:

	GIVEN:		REQUIRED:		
9	$a = 6$,	$c = 12$.	$A = 30^\circ$,	$B = 60^\circ$,	$b = 10.392$.
10	$A = 60^\circ$,	$b = 4$.	$B = 30^\circ$,	$c = 8$,	$a = 6.9282$.
11	$A = 30^\circ$,	$a = 3$.	$B = 60^\circ$,	$c = 6$,	$b = 5.1961$.
12	$a = 4$,	$b = 4$.	$A = B = 45^\circ$,	$c = 5.6568$.	
13	$a = 2$,	$c = 2.82843$.	$A = B = 45^\circ$,	$b = 2$.	
14	$c = 627$,	$A = 23^\circ 30'$.	$B = 66^\circ 30'$,	$a = 250.02$,	$b = 575.0$.
15	$c = 2280$,	$A = 28^\circ 5'$.	$B = 61^\circ 55'$,	$a = 1073.3$,	$b = 2011.6$.
16	$c = 72.15$,	$A = 39^\circ 34'$.	$B = 50^\circ 26'$,	$a = 45.958$,	$b = 55.620$.
17	$c = 1$,	$A = 36^\circ$.	$B = 54^\circ$,	$a = 0.58779$,	$b = 0.80902$.
18	$c = 200$,	$B = 21^\circ 47'$.	$A = 68^\circ 13'$,	$a = 185.73$,	$b = 74.22$.
19	$c = 93.4$,	$B = 76^\circ 25'$.	$A = 13^\circ 35'$,	$a = 21.936$,	$b = 90.788$.
20	$a = 637$,	$A = 4^\circ 35'$.	$B = 85^\circ 25'$,	$b = 7946$,	$c = 7971.5$.
21	$a = 48.532$,	$A = 36^\circ 44'$.	$B = 53^\circ 16'$,	$b = 65.031$,	$c = 81.144$.
22	$a = 0.0008$,	$A = 86^\circ$.	$B = 4^\circ$,	$b = 0.0000559$,	$c = 0.000802$.
23	$b = 50.937$,	$B = 43^\circ 48'$.	$A = 46^\circ 12'$,	$a = 53.116$,	$c = 73.59$.
24	$b = 2$,	$B = 3^\circ 38'$.	$A = 86^\circ 22'$,	$a = 31.497$,	$c = 31.560$.
25	$a = 992$,	$B = 76^\circ 19'$.	$A = 13^\circ 41'$,	$b = 4074.5$.	$c = 4193.6$.
26	$a = 73$,	$B = 68^\circ 52'$.	$A = 21^\circ 8$,	$b = 188.86$,	$c = 202.47$.
27	$a = 2.189$,	$B = 45^\circ 25'$.	$A = 44^\circ 35'$,	$b = 2.2211$,	$c = 3.1185$.
28	$b = 4$,	$A = 37^\circ 56'$.	$B = 52^\circ 4'$,	$a = 3.1176$,	$c = 5.0714$.
29	$c = 8590$,	$a = 4476$.	$A = 31^\circ 24'$,	$B = 58^\circ 36'$,	$b = 7332.8$.
30	$c = 86.53$,	$a = 71.78$.	$A = 56^\circ 3'$,	$B = 33^\circ 57'$,	$b = 48.324$.
31	$c = 9.35$,	$a = 8.49$.	$A = 65^\circ 14'$,	$B = 24^\circ 46'$,	$b = 3.917$.
32	$c = 2194$,	$b = 1312.7$.	$A = 53^\circ 15'$,	$B = 36^\circ 45'$,	$a = 1758$.
33	$c = 30.69$,	$b = 18.256$.	$A = 53^\circ 30'$,	$B = 36^\circ 30'$,	$a = 24.67$.
34	$a = 38.313$,	$b = 19.522$.	$A = 63^\circ$,	$B = 27^\circ$,	$c = 43$.
35	$a = 1.2291$,	$b = 14.950$.	$A = 4^\circ 42'$,	$B = 85^\circ 18'$,	$c = 15$.
36	$a = 415.38$,	$b = 62.080$.	$A = 81^\circ 30'$,	$B = 8^\circ 30'$,	$c = 420$.
37	$a = 13.690$,	$b = 16.926$.	$A = 38^\circ 58'$,	$B = 51^\circ 2'$,	$c = 21.769$.
38	$c = 91.92$,	$a = 2.19$.	$A = 1^\circ 21' 55''$,	$B = 88^\circ 38' 5''$,	$b = 91.894$.

Compute the unknown parts and also the area, having given:

39. $a = 5$, $b = 6$.
40. $a = 0.615$, $c = 70$.
41. $b = \sqrt[3]{2}$, $c = \sqrt{3}$.
42. $a = 7$, $A = 18^\circ 14'$.
43. $b = 12$, $A = 29^\circ 8'$.
44. $c = 68$, $A = 69^\circ 54'$.
45. $c = 27$, $B = 44^\circ 4'$.
46. $a = 47$, $B = 48^\circ 49'$.
47. $b = 9$, $B = 34^\circ 44'$.
48. $c = 8.462$ $B = 86^\circ 4'$.

49. Find the value of F in terms of c and A.
50. Find the value of F in terms of a and A.
51. Find the value of F in terms of b and A.
52. Find the value of F in terms of a and c.
53. Given $F = 58$, $a = 10$; solve the triangle.
54. Given $F = 18$, $b = 5$; solve the triangle.
55. Given $F = 12$, $A = 29°$; solve the triangle.
56. Given $F = 100$, $c = 22$; solve the triangle.

57. Find the angles of a right triangle if the hypotenuse is equal to three times one of the legs.

58. Find the legs of a right triangle if the hypotenuse $= 6$, and one angle is twice the other.

59. In a right triangle given c, and $A = nB$; find a and b.

60. In a right triangle the difference between the hypotenuse and the greater leg is equal to the difference between the two legs; find the angles.

The angle of elevation of an object (or angle of depression, if the object is below the level of the observer) is the angle which a line from the eye to the object makes with a horizontal line in the same vertical plane.

61. At a horizontal distance of 120 feet from the foot of a steeple, the angle of elevation of the top was found to be 60° 30′; find the height of the steeple.

62. From the top of a rock that rises vertically 326 feet out of the water, the angle of depression of a boat was found to be 24°; find the distance of the boat from the foot of the rock.

63. How far is a monument, in a level plain, from the eye, if the height of the monument is 200 feet and the angle of elevation of the top 3° 30′?

64. In order to find the breadth of a river a distance AB was measured along the bank, the point A being directly opposite a tree C on the other side. The angle ABC was also measured. If $AB = 96$ feet, and $ABC = 21° 14'$, find the breadth of the river.

If $ABC = 45°$, what would be the breadth of the river?

65. Find the angle of elevation of the sun when a tower a feet high casts a horizontal shadow b feet long. Find the angle when $a = 120$, $b = 70$.

66. How high is a tree that casts a horizontal shadow b feet in length when the angle of elevation of the sun is $A°$? Find the height of the tree when $b = 80$, $A = 50°$.

67. What is the angle of elevation of an inclined plane if it rises 1 foot in a horizontal distance of 40 feet?

68. A ship is sailing due north-east with a velocity of 10 miles an hour. Find the rate at which she is moving due north, and also due east.

69. In front of a window 20 feet high is a flower-bed 6 feet wide. How long must a ladder be to reach from the edge of the bed to the window?

70. A ladder 40 feet long may be so placed that it will reach a window 33 feet high on one side of the street, and by turning it over without moving its foot it will reach a window 21 feet high on the other side. Find the breadth of the street.

71. From the top of a hill the angles of depression of two successive milestones, on a straight level road leading to the hill, are observed to be 5° and 15°. Find the height of the hill.

72. A fort stands on a horizontal plain. The angle of elevation at a certain point on the plain is 30°, and at a point 100 feet nearer the fort it is 45°. How high is the fort?

73. From a certain point on the ground the angles of elevation of the belfry of a church and of the top of the steeple were found to be 40° and 51° respectively. From a point 300 feet farther off, on a horizontal line, the angle of elevation of the top of the steeple is found to be 33° 45′. Find the distance from the belfry to the top of the steeple.

74. The angle of elevation of the top of an inaccessible fort C, observed from a point A, is 12°. At a point B, 219 feet from A and on a line AB perpendicular to AC, the angle ABC is 61° 45′. Find the height of the fort.

§ 17. THE ISOSCELES TRIANGLE.

An isosceles triangle is divided by the perpendicular from the vertex to the base into two equal right triangles.

Therefore, an isosceles triangle is determined by any two parts that determine one of these right triangles.

Let the parts of an isosceles triangle ABC (Fig. 13), among which the altitude CD is to be included, be denoted as follows:

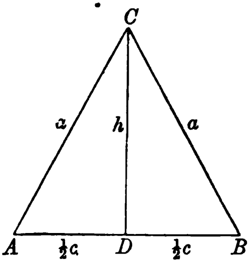

Fig. 13.

$a =$ one of the equal sides.
$c =$ the base.
$h =$ the altitude.
$A =$ one of the equal angles.
$C =$ the angle at the vertex.

For example: Given a and c; required A, C, h.

1. $\cos A = \frac{\frac{1}{2}c}{a} = \frac{c}{2a}$.

2. $C + 2A = 180°$; $\therefore C = 180° - 2A = 2(90° - A)$.

3. h may be found directly in terms of a and c from the equation

$$h^2 + \frac{c^2}{4} = a^2,$$

which gives $$h = \sqrt{(a - \tfrac{1}{2}c)(a + \tfrac{1}{2}c)}.$$

But it is better to find the angles first, and then find h from either one of the two equations,

$$\frac{h}{a} = \sin A, \quad \text{or} \quad \frac{h}{\frac{1}{2}c} = \tan A.$$

whence $$h = a \sin A, \text{ or } \quad h = \tfrac{1}{2}c \tan A.$$

The numerical values of A, C, and h may be computed by the aid of logarithms, as in the case of the right triangle.

The area F of the triangle may be found, when c and h ar given or have been found, by means of the formula

$$F = \tfrac{1}{2}ch.$$

EXERCISE VII.

In an isosceles triangle:

1. Given a and A; find C, c, h.
2. Given a and C; find A, c, h.
3. Given c and A; find C, a, h.
4. Given c and C; find A, a, h.
5. Given h and A; find C, a, c.
6. Given h and C; find A, a, c.
7. Given a and h; find A, C, c.
8. Given c and h; find A, C, a.
9. Given $a = 14.3$, $c = 11$; find A, C, h.
10. Given $a = 0.295$, $A = 68° 10'$; find c, h, F.
11. Given $c = 2.352$, $C = 69° 49'$; find a, h, F.
12. Given $h = 7.4847$, $A = 76° 14'$; find a, c, F.
13. Given $a = 6.71$, $h = 6.60$; find A, C, c.
14. Given $c = 9$, $h = 20$; find A, C, a.
15. Given $c = 147$, $F = 2572.5$; find A, C, a, h.
16. Given $h = 16.8$, $F = 43.68$; find A, C, a, c.
17. Find the value of F in terms of a and c.
18. Find the value of F in terms of a and C.
19. Find the value of F in terms of a and A.
20. Find the value of F in terms of h and C.

21. A barn is 40 × 80 feet, the pitch of the roof is 45°; find the length of the rafters and the area of both sides of the roof.

22. In a unit circle what is the length of the chord corresponding to the angle 45° at the centre?

23. If the radius of a circle $= 30$, and the length of a chord $= 44$, find the angle at the centre.

24. Find the radius of a circle if a chord whose length is 5 subtends at the centre an angle of 133°.

25. What is the angle at the centre of a circle if the corresponding chord is equal to $\frac{2}{3}$ of the radius?

26. Find the area of a circular sector if the radius of the circle $= 12$, and the angle of the sector $= 30°$.

§ 18. THE REGULAR POLYGON.

Lines drawn from the centre of a regular polygon (Fig. 14) to the vertices are radii of the circumscribed circle; and lines drawn from the centre to the middle points of the sides are radii of the inscribed circle. These lines divide the polygon into equal right triangles. Therefore, a regular polygon is determined by a right triangle whose sides are the radius of the circumscribed circle, the radius of the inscribed circle, and half of one side of the polygon.

If the polygon has n sides, the angle of this right triangle at the centre is equal to

$$\frac{1}{2}\left(\frac{360^\circ}{n}\right) \quad \text{or} \quad \frac{180^\circ}{n}.$$

If, also, a side of the polygon, or one of the above-mentioned radii, is given, this triangle may be solved, and the solution gives the unknown parts of the polygon.

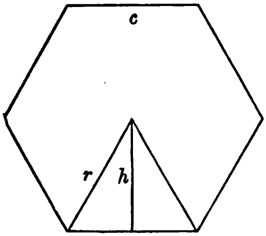

Fig. 14.

Let,

n = number of sides.
c = length of one side.
r = radius of circumscribed circle.
h = radius of inscribed circle.
p = the perimeter.
F = the area.

Then, by Geometry,

$$F = \tfrac{1}{2} hp.$$

EXERCISE VIII.

1. Given $n = 10$, $c = 1$; find r, h, F.
2. Given $n = 12$, $p = 70$; find r, h, F.
3. Given $n = 18$, $r = 1$; find h, p, F.
4. Given $n = 20$, $r = 20$; find h, c, F.
5. Given $n = 8$, $h = 1$; find r, c, F.
6. Given $n = 11$, $F = 20$; find r, h, c.
7. Given $n = 7$, $F = 7$; find r, h, p.

8. Find the side of a regular decagon inscribed in a unit circle.

9. Find the side of a regular decagon circumscribed about a unit circle.

10. If the side of an inscribed regular hexagon is equal to 1, find the side of an inscribed regular dodecagon.

11. Given n and c, and let b denote the side of the inscribed regular polygon having $2n$ sides; find b in terms of n and c.

12. Compute the difference between the areas of a regular octagon and a regular nonagon if the perimeter of each is 16.

13. Compute the difference between the perimeters of a regular pentagon and a regular hexagon if the area of each is 12.

14. From a square whose side is equal to 1 the corners are cut away so that a regular octagon is left. Find the area of this octagon.

15. Find the area of a regular pentagon if its diagonals are each equal to 12.

16. The area of an inscribed regular pentagon is 331.8; find the area of a regular polygon of 11 sides inscribed in the same circle.

17. The perimeter of an equilateral triangle is 20; find the area of the inscribed circle.

18. The area of a regular polygon of 16 sides, inscribed in a circle, is 100; find the area of a regular polygon of 15 sides, inscribed in the same circle.

19. A regular dodecagon is circumscribed about a circle, the circumference of which is equal to 1; find the perimeter of the dodecagon.

20. The area of a regular polygon of 25 sides is equal tc 40; find the area of the ring comprised between the circumferences of the inscribed and the circumscribed circles.

CHAPTER III.

GONIOMETRY.

§ 21. DEFINITION OF GONIOMETRY.

IN order to prepare the way for the solution of an oblique triangle, we now proceed to extend the definitions of the trigonometric functions to angles of all magnitudes, and to deduce certain useful relations of the functions of different angles.

That branch of Trigonometry which treats of trigonometric functions in general, and of their relations, is called **Goniometry.**

§ 22. ANGLES OF ANY MAGNITUDE.

Let the radius OP of a circle (Fig. 16) generate an angle by turning about the centre O. This angle will be measured by the arc described by the point P; and it may have any magnitude, because the arc described by P may have any magnitude.

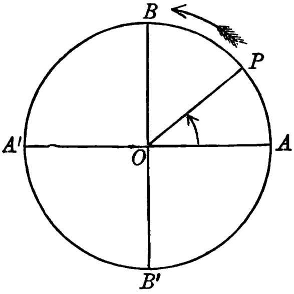

Fig. 16.

Let the horizontal line OA be the initial position of OP, and let OP revolve in the direction shown by the arrow, or opposite to the way clock-hands revolve. Let, also, the four quadrants into which the circle is divided by the horizontal and vertical diameters AA', BB', be numbered I., II., III., IV., in the direction of the motion.

During one revolution OP will form with OA all angles from 0° to 360°. Any particular angle is said to be an angle of the quadrant in which OP lies; so that,

Angles between 0° and 90° are angles of Quadrant I.
Angles between 90° and 180° are angles of Quadrant II.
Angles between 180° and 270° are angles of Quadrant III.
Angles between 270° and 360° are angles of Quadrant IV.

If OP make another revolution, it will describe all angles from 360° to 720°, and so on.

If OP, instead of making another revolution in the direction of the arrow, be supposed to revolve *backwards* about O, this backward motion tends to undo or cancel the original forward motion. Hence, the angle thus generated must be regarded as a *negative* angle; and this negative angle may obviously have any magnitude. Thus we arrive at the conception of an angle of any magnitude, positive or negative.

§ 23. GENERAL DEFINITIONS OF THE FUNCTIONS.

The definitions of the trigonometric functions may be extended to all angles, by making the functions of any angle equal to the line values in a unit circle drawn for the angle in question, as explained in § 3. *But the lines that represent the sine, cosine, tangent, and cotangent must be regarded as negative, if they are opposite in direction to the lines that represent the corresponding functions of an angle in the first quadrant; and the lines that represent the secant and cosecant must be regarded as negative, if they are opposite in direction to the moving radius.*

Figs. 17–20 show the functions drawn for an angle AOP in each quadrant taken in order. In constructing them, it must be remembered that the tangents to the circle are *always* drawn through A and B, *never* through A' or B'.

Let the angle AOP be denoted by x; then, in each figure the *absolute* values of the functions, that is, their values without regard to the signs + and −, are as follows:

$\sin x = MP,$ $\tan x = AT,$ $\sec x = OT.$

$\cos x = OM,$ $\cot x = BS,$ $\csc x = OS.$

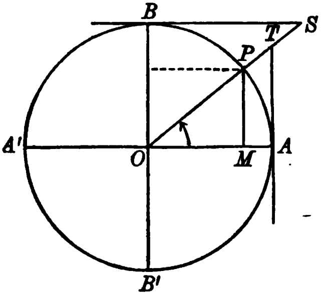

Fig. 17.

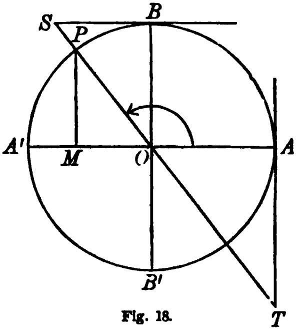

Fig. 18.

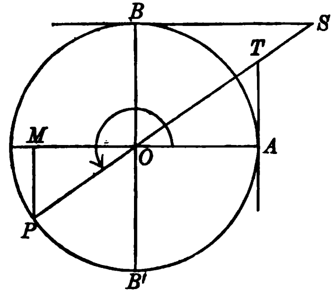

Fig. 19.

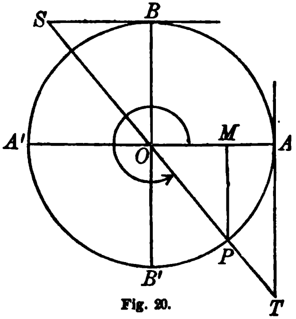

Fig. 20.

Keeping in mind the position of the points A and B, we may define in words the first four functions of the angle x thus:

$\sin x =$ the vertical projection of the moving radius;

$\cos x =$ the horizontal projection of the moving radius;

$\tan x =$ the distance measured along a tangent to the circle from the beginning of the first quadrant to the moving radius produced;

$\cot x =$ the distance measured along a tangent to the circle from the end of the first quadrant to the moving radius produced.

Sec x and csc x are the distances from the centre of the circle measured along the moving radius produced to the tangent and cotangent respectively.

§ 24. ALGEBRAIC SIGNS OF THE FUNCTIONS.

The lengths of the lines, defined above as the functions of any angle, are expressed numerically in terms of the radius of the circle as the unit. But, before these lengths can be treated as algebraic quantities, they must have the sign + or − prefixed, according to the condition stated in § 23.

The reason for this condition lies in that fundamental relation between algebraic and geometric magnitudes, in virtue of which *contrary signs in Algebra correspond to opposite directions in Geometry.*

The sine MP and the tangent AT always extend from the horizontal diameter, but sometimes *upwards* and sometimes *downwards;* the cosine OM and the cotangent BS always extend from the vertical diameter, but sometimes *towards the right* and sometimes *towards the left.* The functions of an angle in the first quadrant are assumed to be positive. Therefore,

1. Sines and tangents extending from the horizontal diameter *upwards*, are positive; *downwards*, negative.
2. Cosines and cotangents extending from the vertical diameter *towards the right*, are positive; *towards the left*, are negative.

The signs of the secant and cosecant are always made to agree with those of the cosine and sine respectively. This agreement is secured if secants and cosecants extending from the centre, *in the direction of the moving radius*, are considered positive; *in the opposite direction*, negative.

Hence, the signs of the functions for each quadrant are:

	I.	II.	III.	IV.
Sine and cosecant	+	+	−	−
Cosine and secant	+	−	−	+
Tangent and cotangent	+	−	+	−

§ 25. Functions of a Variable Angle.

Let the angle x increase continuously from 0° to 360°; what changes will the values of its functions undergo?

It is easy, by reference to Figs. 21–24, to trace these changes throughout all the quadrants.

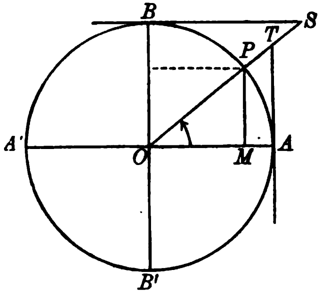

Fig. 21.

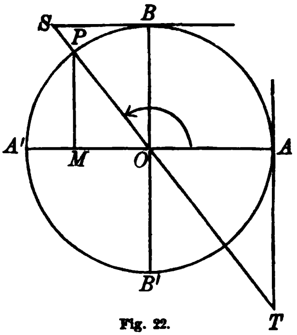

Fig. 22.

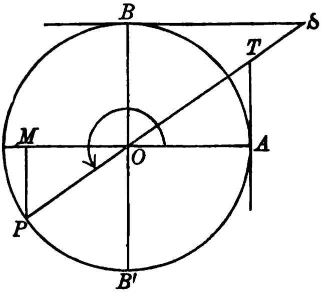

Fig. 23.

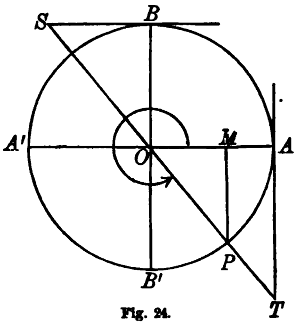

Fig. 24.

1. *The Sine.* In the first quadrant, the sine MP increases from 0 to 1; in the second, it remains positive, and decreases from 1 to 0; in the third, it is negative, and increases in absolute value from 0 to 1; in the fourth, it is negative, and decreases in absolute value from 1 to 0.

2. *The Cosine.* In the first quadrant, the cosine OM decreases from 1 to 0; in the second, it becomes negative and increases in absolute value from 0 to 1; in the third, it is negative and decreases in absolute value from 1 to 0; in the fourth, it is positive and increases from 0 to 1.

3. *The Tangent.* In the first quadrant, the tangent AT increases from 0 to ∞; in the second quadrant, as soon as the angle exceeds 90° by the smallest conceivable amount, the moving radius OP, prolonged in the direction opposite to that of OP, will cut AT at a point T situated very far *below* A; hence, the tangents of angles near 90° in the second quadrant have very large *negative* values. As the angle increases, the tangent AT continues negative but diminishes in absolute value. When $x = 180°$, then T coincides with A, and $\tan 180° = 0$. In the third quadrant, the tangent is positive and increases from 0 to ∞; in the fourth, it is negative and decreases in absolute value from ∞ to 0.

4. *The Cotangent.* In the first quadrant, the cotangent BS decreases from ∞ to 0; in the second quadrant, it is negative and increases in absolute value from 0 to ∞; in the third and fourth quadrants, it has the same sign, and undergoes the same changes as in the first and second quadrants respectively.

5. *The Secant.* In the first quadrant, the secant OT increases from 1 to ∞; in the second quadrant, it becomes negative (being measured in the direction opposite to that of OP), and decreases in absolute value from ∞ to 1, so that $\sec 180° = -1$; in the third quadrant, it continues negative, and increases in absolute value from 1 to ∞; in the fourth quadrant, it is positive, and decreases from ∞ to 1.

6. *The Cosecant.* In the first quadrant, the cosecant OS decreases from ∞ to 1; in the second quadrant, it remains positive, and increases from 1 to ∞; in the third quadrant, it becomes negative, and decreases in absolute value from ∞ to 1, so that $\csc 270° = -1$; in the fourth quadrant, it is negative, and increases in absolute value from 1 to ∞.

The limiting values of the functions are as follows:

	0°	90°	180°	270°	360°
Sine	± 0	1	± 0	− 1	± 0
Cosine	1	± 0	− 1	± 0	1
Tangent	± 0	± ∞	± 0	± ∞	± 0
Cotangent	± ∞	± 0	± ∞	± 0	± ∞
Secant	1	± ∞	− 1	± ∞	1
Cosecant	± ∞	1	± ∞	− 1	± ∞

Sines and cosines extend from $+1$ to -1; tangents and cotangents from $+\infty$ to $-\infty$; secants and cosecants from $+\infty$ to $+1$, and from -1 to $-\infty$.

In the table given above the double sign ± is placed before 0 and ∞. From the preceding investigation it appears that the functions *always change sign in passing through* 0 *and* ∞; and the sign + or − prefixed to 0 or ∞ simply shows the direction from which the value is reached.

Take, for example, tan 90°: The nearer an acute angle is to 90°, the greater the *positive* value of its tangent; and the nearer an obtuse angle is to 90°, the greater the *negative* value of its tangent. When the angle *is* 90°, OP (Fig. 21) is parallel to AT, and cannot meet it. But tan 90° may be regarded as extending either in the positive or in the negative direction; and according to the view taken, it will be $+\infty$ or $-\infty$.

§ 26. FUNCTIONS OF ANGLES LARGER THAN 360°.

It is obvious that the functions of $360° + x$ are the same both in sign and in absolute value as those of x; for the moving radius has the same position in both cases. In general, if n denote any positive whole number,

The functions of $(n \times 360° + x)$ *are the same as those of* x.

For example: the functions of $2200°$ = the functions of $(6 \times 360° + 40°)$ = the functions of 40°.

§ 27. EXTENSION OF FORMULAS [1]–[3] TO ALL ANGLES.

The Formulas established for *acute* angles in § 5 hold true for *all* angles. Thus, Formula [1],

$$\sin^2 x + \cos^2 x = 1,$$

is universally true; for, whether MP and OM (Figs. 21–24) are positive or negative, $\overline{MP}^2$ and $\overline{OM}^2$ are always positive, and in each quadrant $\overline{MP}^2 + \overline{OM}^2 = \overline{OP}^2 = 1$.

Also, Formulas

$$[2] \quad \tan x = \frac{\sin x}{\cos x},$$

and

$$[3] \begin{cases} \sin x \times \csc x = 1, \\ \cos x \times \sec x = 1, \\ \tan x \times \cot x = 1, \end{cases}$$

are universally true; for the algebraic signs of the functions, as given in the table at the end of § 24, agree with those in Formulas [2] and [3]; and with regard to the absolute values, we have in each quadrant from the similar triangles OMP, OAT, OBS, (Figs. 21–24) the proportions

$$\begin{aligned} AT : OA &= MP : OM, \\ MP : OP &= OB : OS, \\ OM : OP &= OA : OT, \\ AT : OA &= OB : BS, \end{aligned}$$

which, by substituting 1 for the radius, and the right names for the other lines, are easily reduced to the above formulas.

Formulas [1]–[3] enable us, from a given value of one function, to find the absolute values of the other five functions, and also the sign of the reciprocal function. But in order to determine the proper signs to be placed before the other four functions, we must know the quadrant to which the angle in question belongs; or what amounts to the same thing, the sign of any *one* of these four functions; for, by reference to the Table of Signs (§ 24) it will be seen that the signs of any two functions that are not reciprocals determine the quadrant to which the angle belongs.

Example. Given $\sin x = +\frac{4}{5}$, and $\tan x$ negative; find the values of the other functions.

Since $\sin x$ is positive, x must be an angle in Quadrant I. or in Quadrant II.; but, since also $\tan x$ is negative, Quadrant I. is inadmissable.

By [1], $\cos x = \pm\sqrt{1 - \frac{16}{25}} = \pm \frac{3}{5}$.

Since the angle is in Quadrant II. the minus sign must be taken, and we have

$$\cos x = -\frac{3}{5}.$$

By [2] and [3],

$$\tan x = -\frac{4}{3},\quad \cot x = -\frac{3}{4},\quad \sec x = -\frac{5}{3},\quad \csc x = \frac{5}{4}.$$

Exercise IX.

1. Construct the functions of an angle in Quadrant II. What are their signs?

2. Construct the functions of an angle in Quadrant III. What are their signs?

3. Construct the functions of an angle in Quadrant IV. What are their signs?

4. What are the signs of the functions of the following angles: 340°, 239°, 145°, 400°, 700°, 1200°, 3800°?

5. How many angles less than 360° have the value of the sine equal to $+\frac{5}{7}$, and in what quadrants do they lie?

6. How many values less than 720° can the angle x have if $\cos x = +\frac{2}{3}$, and in what quadrants do they lie?

7. If we take into account only angles less than 180°, how many values can x have if $\sin x = \frac{4}{7}$? if $\cos x = \frac{1}{5}$? if $\cos x = -\frac{4}{5}$? if $\tan x = \frac{2}{3}$? if $\cot x = -7$?

8. Within what limits must the angle x lie if $\cos x = -\frac{2}{9}$? if $\cot x = 4$? if $\sec x = 80$? if $\csc x = -3$? (x to be less than 360°.)

9. In what quadrant does an angle lie if sine and cosine are both negative? if cosine and tangent are both negative? if the cotangent is positive and the sine negative?

10. Between 0° and 3600° how many angles are there whose sines have the absolute value $\frac{3}{5}$? Of these sines how many are positive and how many negative?

11. In finding $\cos x$ by means of the equation $\cos x = \pm\sqrt{1 - \sin^2 x}$, when must we choose the positive sign and when the negative sign?

12. Given $\cos x = -\sqrt{\frac{1}{2}}$; find the other functions when x is an angle in Quadrant II.

13. Given $\tan x = \sqrt{3}$; find the other functions when x is an angle in Quadrant III.

14. Given $\sec x = +7$, and $\tan x$ negative; find the other functions of x.

15. Given $\cot x = -3$; find all the possible values of the other functions.

16. What functions of an angle of a triangle may be negative? In what case are they negative?

17. What functions of an angle of a triangle determine the angle, and what functions fail to do so?

18. Why may $\cot 360°$ be considered equal either to $+\infty$ or to $-\infty$?

19. Obtain by means of Formulas [1]–[3] the other functions of the angles given:

(i.) $\tan 90° = \infty$. (iii.) $\cot 270° = 0$.
(ii.) $\cos 180° = -1$. (iv.) $\csc 360° = -\infty$.

20. Find the values of $\sin 450°$, $\tan 540°$, $\cos 630°$, $\cot 720°$, $\sin 810°$, $\csc 900°$.

21. For what angle in each quadrant are the absolute values of the sine and cosine alike?

Compute the values of the following expressions:

22. $a \sin 0° + b \cos 90° - c \tan 180°$.

23. $a \cos 90° - b \tan 180° + c \cot 90°$.

24. $a \sin 90° - b \cos 360° + (a - b) \cos 180°$.

25. $(a^2 - b^2) \cos 360° - 4ab \sin 270°$.

§ 28. REDUCTION OF FUNCTIONS TO THE FIRST QUADRANT.

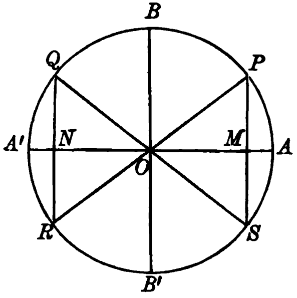

Fig. 25.

In a unit circle (Fig. 25) draw two diameters *PR* and *QS* equally inclined to the horizontal diameter *AA'*, or so that the angles *AOP*, *A'OQ*, *A'OR*, and *AOS* shall be equal. From the points *P*, *Q*, *R*, *S* let fall perpendiculars to *AA'*; the four right triangles thus formed, with a common vertex at *O*, are equal; because they have equal hypotenuses (radii of the circle) and equal acute angles at *O*. Therefore, the perpendiculars *PM*, *QN*, *RN*, *SM*, are equal. Now these four lines are the sines of the angles *AOP*, *AOQ*, *AOR*, and *AOS*, respectively. Therefore, *in absolute value*,

$$\sin AOP = \sin AOQ = \sin AOR = \sin AOS.$$

And from § 27 it follows that *in absolute value* the cosines of these angles are also equal; and likewise the tangents, the cotangents, the secants, and the cosecants.*

Hence, *for every acute angle* (*AOP*) *there is an angle in each of the higher quadrants whose functions, in absolute value, are equal to those of this acute angle.*

Let $\angle AOP = x$, $\angle POB = y$; then $x + y = 90°$, and the functions of x are equal to the co-named functions of y (§ 4); further,

$$\begin{aligned}
&\angle AOQ \text{ (in Quadrant II.)} &&= 180° - x = 90° + y,\\
&\angle AOR \text{ (in Quadrant III.)} &&= 180° + x = 270° - y,\\
&\angle AOS \text{ (in Quadrant IV.)} &&= 360° - x = 270° + y.
\end{aligned}$$

Hence, if we prefix in each case the proper sign (§ 24), we have the two following series of Formulas:

---

* In future, secants and cosecants will be disregarded. They may be found by [3] if wanted, but are seldom used in computations.

Angle in Quadrant II.

$$\begin{array}{ll}
\sin(180^\circ - x) = \sin x. & \sin(90^\circ + y) = \cos y. \\
\cos(180^\circ - x) = -\cos x. & \cos(90^\circ + y) = -\sin y. \\
\tan(180^\circ - x) = -\tan x. & \tan(90^\circ + y) = -\cot y. \\
\cot(180^\circ - x) = -\cot x. & \cot(90^\circ + y) = -\tan y.
\end{array}$$

Angle in Quadrant III.

$$\begin{array}{ll}
\sin(180^\circ + x) = -\sin x. & \sin(270^\circ - y) = -\cos y. \\
\cos(180^\circ + x) = -\cos x. & \cos(270^\circ - y) = -\sin y. \\
\tan(180^\circ + x) = \tan x. & \tan(270^\circ - y) = \cot y. \\
\cot(180^\circ + x) = \cot x. & \cot(270^\circ - y) = \tan y.
\end{array}$$

Angle in Quadrant IV.

$$\begin{array}{ll}
\sin(360^\circ - x) = -\sin x. & \sin(270^\circ + y) = -\cos y. \\
\cos(360^\circ - x) = \cos x. & \cos(270^\circ + y) = \sin y. \\
\tan(360^\circ - x) = -\tan x. & \tan(270^\circ + y) = -\cot y. \\
\cot(360^\circ - x) = -\cot x. & \cot(270^\circ + y) = -\tan y.
\end{array}$$

Hence, by selecting the right formulas,

The functions of all angles can be reduced to the functions of angles not greater than 45°. Thus, to find the functions of 220° and 230°, we should consider 220° as $(180^\circ + 40^\circ)$, but 230° as $(270^\circ - 40^\circ)$.

It is evident from these formulas that,

If an acute angle be added to or subtracted from **180° or 360°**, *the functions of the resulting angle are equal in absolute value to the* **like-named** *functions of the acute angle; but if an acute angle be added to or subtracted from* **90° or 270°**, *the functions of the resulting angle are equal in absolute value to the* **co-named** *functions of the acute angle.*

It is evident from the formulas for $(180^\circ - x)$ that,

A given value of a sine determines two supplementary angles, one acute, the other obtuse; a given value of any other function (except the cosecant) determines only one angle: acute if the value is positive, obtuse if the value is negative.

§ 29. ANGLES WHOSE DIFFERENCE IS 90°.

The general form of two such angles is x and $90° + x$, and they must lie in adjoining quadrants. The relations between their functions were found in § 28, but only for the case when x is acute. These relations, however, may be shown to hold true for all values of x.

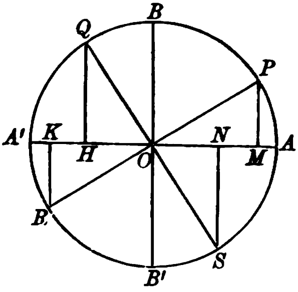

Fig. 26.

In a unit circle (Fig. 26) draw two diameters PR and QS perpendicular to each other, and let fall to AA' the perpendiculars PM, QH, RK, and SN. The right triangles OMP, OHQ, OKR, and ONS are equal, because they have equal hypotenuses and equal acute angles POM, OQH, ROK, and OSN.

Therefore, $\quad OM = QH = OK = NS,$
and $\quad PM = OH = KR = ON.$

Hence, taking also into account the algebraic sign,

$$\begin{array}{llll}
\sin AOQ = & \cos AOP; & \sin AOS & = \cos AOR; \\
\cos AOQ = & -\sin AOP; & \cos AOS & = -\sin AOR; \\
\sin AOR = & \cos AOQ; & \sin(360° + AOP) & = \cos AOS; \\
\cos AOR = & -\sin AOQ; & \cos(360° + AOP) & = -\sin AOS.
\end{array}$$

In all these equations, if x denote the angle on the right-hand side, the angle on the left-hand side will be $90° + x$. Therefore, if x be an angle in any one of the four quadrants,

$$\sin(90° + x) = \cos x,$$
$$\cos(90° + x) = -\sin x.$$

And, by § 27,

$$\tan(90° + x) = -\cot x,$$
$$\cot(90° + x) = -\tan x.$$

In like manner, it can be shown that all the formulas of § 28 hold true, whatever be the value of the angle x.

§ 30. FUNCTIONS OF A NEGATIVE ANGLE.

If the angle AOP (Fig. 25) is denoted by x, the equal angle AOS, generated by a *backward* rotation of the moving radius from the initial position OA, will be denoted by $-x$. It is obvious that the position OS of the moving radius for this angle is identical with its position for the angle $360° - x$. Therefore, the functions of the angle $-x$ are the same as those of the angle $360° - x$; or (§ 28),

$$\sin(-x) = -\sin x, \qquad \tan(-x) = -\tan x,$$
$$\cos(-x) = \cos x, \qquad \cot(-x) = -\cot x.$$

EXERCISE X.

1. Express $\sin 250°$ in terms of the functions of an acute angle greater than 45°, and also in terms of the functions of an acute angle less than 45°.

Ans. 1. $\sin 250° = \sin(180° + 70°) = -\sin 70°$.
2. $\sin 250° = \sin(270° - 20°) = -\cos 20°$.

Express the following functions in terms of the functions of angles less than 45°:

2. $\sin 172°$.
3. $\cos 100°$.
4. $\tan 125°$.
5. $\cot 91°$.
6. $\sec 110°$.
7. $\csc 157°$.
8. $\sin 204°$.
9. $\cos 359°$.
10. $\tan 300°$.
11. $\cot 264°$.
12. $\sec 244°$.
13. $\csc 271°$.
14. $\sin 163° 49'$.
15. $\cos 195° 33'$.
16. $\tan 269° 15'$.
17. $\cot 139° 17'$.
18. $\sec 299° 45'$.
19. $\csc 92° 25'$.

Express all the functions of the following negative angles in terms of those of positive angles less than 45°:

20. $-75°$.
21. $-127°$.
22. $-200°$.
23. $-345°$.
24. $-52° 37'$.
25. $-196° 54'$.
26. Find the functions of 120°.

HINT. $120° = 180° - 60°$, or, $120° = 90° + 30°$; then apply § 28.

Find the functions of the following angles:

27. 135°.
28. 150°.
29. 210°.
30. 225°.
31. 240°.
32. 300°.
33. $-30°$.
34. $-225°$.

35. Given $\sin x = -\sqrt{\frac{1}{2}}$, and $\cos x$ negative; find the other functions of x, and the value of x.

36. Given $\cot x = -\sqrt{3}$, and x in Quadrant II.; find the other functions of x, and the value of x.

37. Find the functions of 3540°.

38. What angles less than 360° have a sine equal to $-\frac{1}{2}$? a tangent equal to $-\sqrt{3}$?

39. Which of the angles mentioned in Examples 27–34 have a cosine equal to $-\sqrt{\frac{1}{2}}$? a cotangent equal to $-\sqrt{3}$?

40. What values of x between 0° and 720° will satisfy the equation $\sin x = +\frac{1}{2}$?

41. In each of the following cases find the other angle between 0° and 360° for which the corresponding function (sign included) has the same value: $\sin 12°$, $\cos 26°$, $\tan 45°$, $\cot 72°$; $\sin 191°$, $\cos 120°$, $\tan 244°$, $\cot 357°$.

42. Given $\tan 238° = 1.6$; find $\sin 122°$.

43. Given $\cos 333° = 0.89$; find $\tan 117°$.

Simplify the following expressions:

44. $a \cos(90° - x) + b \cos(90° + x)$.

45. $m \cos(90° - x) \sin(90° - x)$.

46. $(a - b) \tan(90° - x) + (a + b) \cot(90° + x)$.

47. $a^2 + b^2 - 2ab \cos(180° - x)$.

48. $\sin(90° + x) \sin(180° + x) + \cos(90° + x) \cos(180° - x)$.

49. $\cos(180° + x) \cos(270° - y) - \sin(180° + x) \sin(270° - y)$.

50. $\tan x + \tan(-y) - \tan(180° - y)$.

51. For what values of x is the expression $\sin x + \cos x$ positive, and for what values negative? Represent the result by a drawing in which the sectors corresponding to the negative values are shaded.

52. Answer the question of last example for $\sin x - \cos x$.

53. Find the functions of $(x - 90°)$ in terms of the functions of x.

54. Find the functions of $(x - 180°)$ in terms of the functions of x.

§ 31. Functions of the Sum of Two Angles.

In a unit circle (Fig. 27) let the angle $AOB = x$, the angle $BOC = y$; then the angle $AOC = x + y$.

In order to express $\sin(x+y)$ and $\cos(x+y)$ in terms of the sines and cosines of x and y, draw $CF \perp OA$, $CD \perp OB$, $DE \perp OA$, $DG \perp CF$; then $CD = \sin y$, $OD = \cos y$, and the angle $DCG =$ the angle $GDO = x$. Also,

$\sin(x+y) = CF = DE + CG.$

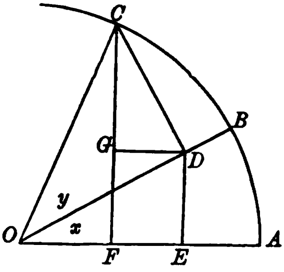

Fig. 27.

$$\frac{DE}{OD} = \sin x; \quad \text{hence, } DE = \sin x \times OD = \sin x \cos y.$$

$$\frac{CG}{CD} = \cos x; \quad \text{hence, } CG = \cos x \times CD = \cos x \sin y.$$

Therefore, $\mathbf{\sin(x+y) = \sin x \cos y + \cos x \sin y.}$ [4]

Again, $\cos(x+y) = OF = OE - DG.$

$$\frac{OE}{OD} = \cos x; \quad \text{hence, } OE = \cos x \times OD = \cos x \cos y.$$

$$\frac{DG}{CD} = \sin x; \quad \text{hence, } DG = \sin x \times CD = \sin x \sin y.$$

Therefore, $\mathbf{\cos(x+y) = \cos x \cos y - \sin x \sin y.}$ [5]

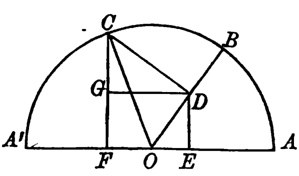

Fig. 28.

In this proof x and y, and also the sum $x + y$, are assumed to be acute angles. If the sum $x+y$ of the acute angles x and y is obtuse, as in Fig. 28, the proof remains, word for word, the same as above, the only difference being that the sign of OF will be negative, as DG is now greater than OE. The above formulas, therefore, hold true for all acute angles x and y.

If these formulas hold true for any two acute angles x and y, they hold true when one of the angles is increased by 90°. Thus, if for x we write $x' = 90° + x$, then, by § 29,

$$\begin{aligned}\sin(x'+y) &= \sin(90°+x+y) = \cos(x+y),\\ \cos(x'+y) &= \cos(90°+x+y) = -\sin(x+y).\end{aligned}$$

Hence, by [5], $\sin(x'+y) = \cos x \cos y - \sin x \sin y$,
by [4], $\cos(x'+y) = -\sin x \cos y - \cos x \sin y$.

Now, by § 29, $\cos x = \sin(90°+x) = \sin x'$,
$\sin x = -\cos(90°+x) = -\cos x'$.

Therefore, by putting $\sin x'$ for $\cos x$, and $-\cos x'$ for $\sin x$, in the right-hand members of the above equations,

$$\begin{aligned}\sin(x'+y) &= \sin x' \cos y + \cos x' \sin y,\\ \cos(x'+y) &= \cos x' \cos y - \sin x' \sin y.\end{aligned}$$

Hence, it follows that Formulas [4] and [5] are universally true. For they have been proved true for any two acute angles, and also true when one of these angles is increased by 90°; hence they are true for each repeated increase of one or the other angle by 90°, and therefore true for the sum of any two angles whatever.

By § 27,

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

If we divide each term of the numerator and denominator of the last fraction by $\cos x \cos y$, and again apply § 27, we obtain

$$\tan(\mathbf{x}+\mathbf{y}) = \frac{\tan \mathbf{x} + \tan \mathbf{y}}{1 - \tan \mathbf{x} \tan \mathbf{y}}. \qquad [6]$$

In like manner, by dividing each term of the numerator and denominator of the value of $\cot(x+y)$ by $\sin x \sin y$, we obtain

$$\cot(\mathbf{x}+\mathbf{y}) = \frac{\cot \mathbf{x} \cot \mathbf{y} - 1}{\cot \mathbf{x} + \cot \mathbf{y}}. \qquad [7]$$

§ 32. FUNCTIONS OF THE DIFFERENCE OF TWO ANGLES.

In a unit circle (Fig. 29) let the angle $AOB = x$, the angle $COB = y$; then the angle $AOC = x - y$.

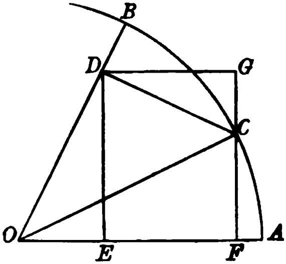

Fig. 29.

In order to express $\sin(x-y)$ and $\cos(x-y)$ in terms of the sines and cosines of x and y, draw $CF \perp OA$, $CD \perp OB$, $DE \perp OA$, $DG \perp FC$ prolonged; then $CD = \sin y$, $OD = \cos y$, and the angle $DCG =$ the angle $EDC = x$. And, $\sin(x-y) = CF = DE - CG$.

$$\frac{DE}{OD} = \sin x; \quad \text{hence, } DE = \sin x \times OD = \sin x \cos y.$$

$$\frac{CG}{CD} = \cos x; \quad \text{hence, } CG = \cos x \times CD = \cos x \sin y.$$

Therefore, $\mathbf{\sin(x-y) = \sin x \cos y - \cos x \sin y.}$ [8]

Again, $\cos(x-y) = OF = OE + DG.$

$$\frac{OE}{OD} = \cos x; \quad \text{hence, } OE = \cos x \times OD = \cos x \cos y.$$

$$\frac{DG}{CD} = \sin x; \quad \text{hence, } DG = \sin x \times CD = \sin x \sin y.$$

Therefore, $\mathbf{\cos(x-y) = \cos x \cos y + \sin x \sin y.}$ [9]

In this proof, both x and y are assumed to be acute angles; but, whatever be the values of x and y, the same method of proof will always lead to Formulas [8] and [9], when due regard is paid to the algebraic signs.

The general application of these formulas may be at once shown by deducing them from the general formulas established in § 31, as follows:

It is obvious that $(x-y)+y = x$. If we apply Formulas [4] and [5] to $(x-y)+y$, then

$$\sin\{(x-y)+y\} \text{ or } \sin x = \sin(x-y)\cos y + \cos(x-y)\sin y,$$
$$\cos\{(x-y)+y\} \text{ or } \cos x = \cos(x-y)\cos y - \sin(x-y)\sin y.$$

Multiply the first equation by $\cos y$, the second by $\sin y$,

$$\sin x \cos y = \sin(x-y)\cos^2 y + \cos(x-y)\sin y \cos y,$$
$$\cos x \sin y = -\sin(x-y)\sin^2 y + \cos(x-y)\sin y \cos y;$$

whence, by subtraction,

$$\sin x \cos y - \cos x \sin y = \sin(x-y)(\sin^2 y + \cos^2 y).$$

But $\sin^2 y + \cos^2 y = 1$; therefore, by transposing,

$$\sin(x-y) = \sin x \cos y - \cos x \sin y.$$

Again, if we multiply the first equation by $\sin y$, the second equation by $\cos y$, and add the results, we obtain, by reducing,

$$\cos(x-y) = \cos x \cos y + \sin x \sin y.$$

Therefore, Formulas [8] and [9], like [4] and [5], from which they have been derived, are universally true.

From [8] and [9], by proceeding as in § 31, we obtain

$$\tan(\mathbf{x}-\mathbf{y}) = \frac{\tan \mathbf{x} - \tan \mathbf{y}}{1 + \tan \mathbf{x} \tan \mathbf{y}}. \qquad [10]$$

$$\cot(\mathbf{x}-\mathbf{y}) = \frac{\cot \mathbf{x} \cot \mathbf{y} + 1}{\cot \mathbf{y} - \cot \mathbf{x}}. \qquad [11]$$

Formulas [4]–[11] may be combined as follows:

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$

$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}.$$

§ 33. FUNCTIONS OF TWICE AN ANGLE.

If, in Formulas [4]–[7], $y = x$, they become:

$$\sin 2x = 2 \sin x \cos x. \quad [12] \qquad \cos 2x = \cos^2 x - \sin^2 x. \quad [13]$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}. \quad [14] \qquad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x}. \quad [15]$$

By these formulas the functions of twice an angle are found when the functions of the angle are given.

§ 34. FUNCTIONS OF HALF AN ANGLE.

Take the formulas

$$\begin{array}{lrcll} & \cos^2 x + \sin^2 x & = & 1 & [1] \\ & \cos^2 x - \sin^2 x & = & \cos 2x & [13] \\ \hline \text{Subtract,} & 2 \sin^2 x & = & 1 - \cos 2x & \\ \text{Add,} & 2 \cos^2 x & = & 1 + \cos 2x & \end{array}$$

Whence

$$\sin x = \pm \sqrt{\frac{1 - \cos 2x}{2}}, \qquad \cos x = \pm \sqrt{\frac{1 + \cos 2x}{2}}.$$

These values, if z is put for $2x$, and hence $\frac{1}{2} z$ for x, become

$$\sin \tfrac{1}{2} z = \pm \sqrt{\frac{1 - \cos z}{2}}. \quad [16] \qquad \cos \tfrac{1}{2} z = \pm \sqrt{\frac{1 + \cos z}{2}}. \quad [17]$$

Hence, by division (§ 27),

$$\tan \tfrac{1}{2} z = \pm \sqrt{\frac{1 - \cos z}{1 + \cos z}}. \quad [18] \qquad \cot \tfrac{1}{2} z = \pm \sqrt{\frac{1 + \cos z}{1 - \cos z}}. \quad [19]$$

By these formulas the functions of half an angle may be computed when the cosine of the entire angle is given.

The proper sign to be placed before the root in each case depends on the quadrant in which the angle $\frac{1}{2} z$ lies. (§ 24.)

Let the student show from Formula [18] that

$$\tan \tfrac{1}{2} B = \sqrt{\frac{c - a}{c + a}}. \quad \text{(See page 19, Note 2.)}$$

§ 35. Sums and Differences of Functions.

From [4], [5], [8], and [9], by addition and subtraction:

$$\begin{aligned}\sin(x+y)+\sin(x-y)&=2\sin x\cos y,\\ \sin(x+y)-\sin(x-y)&=2\cos x\sin y,\\ \cos(x+y)+\cos(x-y)&=2\cos x\cos y,\\ \cos(x+y)-\cos(x-y)&=-2\sin x\sin y;\end{aligned}$$

or, by making $x+y=A$, and $x-y=B$,

and therefore, $x=\frac{1}{2}(A+B)$, and $y=\frac{1}{2}(A-B)$,

$$\mathbf{\sin A+\sin B=2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B).} \quad [20]$$

$$\mathbf{\sin A-\sin B=2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).} \quad [21]$$

$$\mathbf{\cos A+\cos B=2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B).} \quad [22]$$

$$\mathbf{\cos A-\cos B=-2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).} \quad [23]$$

From [20] and [21], by division, we obtain

$$\frac{\sin A+\sin B}{\sin A-\sin B}=\tan\tfrac{1}{2}(A+B)\cot\tfrac{1}{2}(A-B);$$

or, since $\cot\frac{1}{2}(A-B)=\dfrac{1}{\tan\frac{1}{2}(A-B)}$,

$$\mathbf{\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}.} \quad [24]$$

Exercise XI.

1. Find the value of $\sin(x+y)$ and $\cos(x+y)$, when $\sin x=\frac{3}{5}$, $\cos x=\frac{4}{5}$, $\sin y=\frac{5}{13}$, $\cos y=\frac{12}{13}$.

2. Find $\sin(90°-y)$ and $\cos(90°-y)$ by making $x=90°$ in Formulas [8] and [9].

Find, by Formulas [4]–[11], the first four functions of:

3. $90°+y$.
4. $180°-y$.
5. $180°+y$.
6. $270°-y$.
7. $270°+y$.
8. $360°-y$.
9. $360°+y$.
10. $x-90°$.
11. $x-180°$.
12. $x-270°$.
13. $-y$.
14. $45°-y$.
15. $45°+y$.
16. $30°+y$.
17. $60°-y$.

18. Find $\sin 3x$ in terms of $\sin x$.

19. Find $\cos 3x$ in terms of $\cos x$.

20. Given $\tan \frac{1}{2}x = 1$; find $\cos x$.

21. Given $\cot \frac{1}{2}x = \sqrt{3}$; find $\sin x$.

22. Given $\sin x = 0.2$; find $\sin \frac{1}{2}x$ and $\cos \frac{1}{2}x$.

23. Given $\cos x = 0.5$; find $\cos 2x$ and $\tan 2x$.

24. Given $\tan 45° = 1$; find the functions of $22° 30'$.

25. Given $\sin 30° = 0.5$; find the functions of $15°$.

26. Prove that $\tan 18° = \dfrac{\sin 33° + \sin 3°}{\cos 33° + \cos 3°}$.

Prove the following formulas:

27. $\sin 2x = \dfrac{2 \tan x}{1 + \tan^2 x}$.

28. $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$.

29. $\tan \frac{1}{2}x = \dfrac{\sin x}{1 + \cos x}$.

30. $\cot \frac{1}{2}x = \dfrac{\sin x}{1 - \cos x}$.

31. $\sin \frac{1}{2}x \pm \cos \frac{1}{2}x = \sqrt{1 \pm \sin x}$.

32. $\dfrac{\tan x \pm \tan y}{\cot x \pm \cot y} = \pm \tan x \tan y$.

33. $\tan(45° - x) = \dfrac{1 - \tan x}{1 + \tan x}$.

If A, B, C are the angles of a triangle, prove that:

34. $\sin A + \sin B + \sin C = 4 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C$.

35. $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C$.

36. $\tan A + \tan B + \tan C = \tan A \times \tan B \times \tan C$.

37. $\cot \frac{1}{2} A + \cot \frac{1}{2} B + \cot \frac{1}{2} C = \cot \frac{1}{2} A \times \cot \frac{1}{2} B \times \cot \frac{1}{2} C$.

Change to forms more convenient for logarithmic computation:

38. $\cot x + \tan x$.

39. $\cot x - \tan x$.

40. $\cot x + \tan y$.

41. $\cot x - \tan y$.

42. $\dfrac{1 - \cos 2x}{1 + \cos 2x}$.

43. $1 + \tan x \tan y$.

44. $1 - \tan x \tan y$.

45. $\cot x \cot y + 1$.

46. $\cot x \cot y - 1$.

47. $\dfrac{\tan x + \tan y}{\cot x + \cot y}$.

CHAPTER IV.

THE OBLIQUE TRIANGLE.

§ 36. LAW OF SINES.

Let A, B, C denote the angles of a triangle ABC (Figs. 30 and 31), and a, b, c, respectively, the lengths of the opposite sides.

Draw $CD \perp AB$, and meeting AB (Fig. 30) or AB produced (Fig. 31) at D. Let $CD = h$.

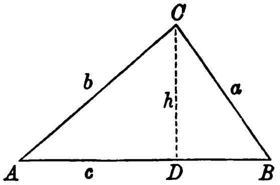

Fig. 30.

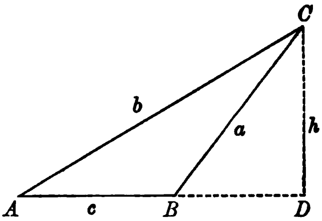

Fig. 31.

In both figures, $\dfrac{h}{b} = \sin A.$

In Fig. 30, $\dfrac{h}{a} = \sin B.$

In Fig. 31, $\dfrac{h}{a} = \sin(180° - B) = \sin B.$

Therefore, whether h lies within or without the triangle, we obtain, by division,

$$\frac{\mathrm{a}}{\mathrm{b}} = \frac{\sin \mathrm{A}}{\sin \mathrm{B}}. \qquad [25]$$

By drawing perpendiculars from the vertices A and B to the opposite sides we may obtain, in the same way,

$$\frac{b}{c}=\frac{\sin B}{\sin C}, \qquad \frac{a}{c}=\frac{\sin A}{\sin C}.$$

Hence the Law of Sines, which may be thus stated:

The sides of a triangle are proportional to the sines of the opposite angles.

If we regard these three equations as proportions, and take them by alternation, it will be evident that they may be written in the symmetrical form,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

Each of these equal ratios has a simple geometrical meaning which will appear if the Law of Sines is proved as follows:

Circumscribe a circle about the triangle ABC (Fig. 32), and draw the radii OA, OB, OC; these radii divide the triangle into three isosceles triangles. Let R denote the radius. Draw $OM \perp BC$. By Geometry, the angle $BOC = 2A$; hence, the angle $BOM = A$, then $BM = R \sin BOM = R \sin A$.

$\therefore BC$ or $a = 2R \sin A$.

In like manner, $b = 2R \sin B$, and $c = 2R \sin C$. Whence we obtain

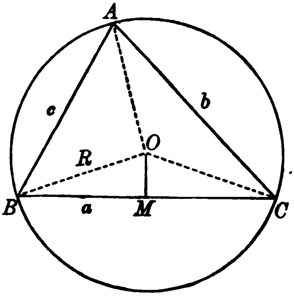

Fig. 32.

$$2R=\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

That is: *The ratio of any side of a triangle to the sine of the opposite angle is numerically equal to the diameter of the circumscribed circle.*

§ 37. Law of Cosines.

This law gives the value of one side of a triangle in terms of the other two sides and the angle included between them.

In Figs. 30 and 31, $a^2 = h^2 + BD^2$.

In Fig. 30, $BD = c - AD$;

in Fig. 31, $BD = AD - c$;

in both cases, $BD^2 = AD^2 - 2c \times AD + c^2$.

Therefore, in all cases, $a^2 = h^2 + AD^2 + c^2 - 2c \times AD$.

Now, $h^2 + AD^2 = b^2$,

and $AD = b \cos A$.

Therefore,

$$\mathbf{a^2 = b^2 + c^2 - 2bc \cos A.} \qquad [26]$$

In like manner, it may be proved that

$$b^2 = a^2 + c^2 - 2ac \cos B,$$
$$c^2 = a^2 + b^2 - 2ab \cos C.$$

The three formulas have precisely the same form, and the law may be stated as follows:

The square of any side of a triangle is equal to the sum of the squares of the other two sides, diminished by twice the product of the sides and the cosine of the included angle.

§ 38. Law of Tangents.

By § 36, $a : b = \sin A : \sin B$;

whence, by the Theory of Proportion,

$$\frac{a - b}{a + b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

But by [24], page 48,

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}.$$

Therefore,

$$\mathbf{\frac{a - b}{a + b} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}.} \qquad [27]$$

By merely changing the letters,

$$\frac{a-c}{a+c}=\frac{\tan\frac{1}{2}(A-C)}{\tan\frac{1}{2}(A+C)}, \quad \frac{b-c}{b+c}=\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}.$$

Hence the Law of Tangents:

The difference of two sides of a triangle is to their sum as the tangent of half the difference of the opposite angles is to the tangent of half their sum.

NOTE. If in [27] $b>a$, then $B>A$. The formula is still true, but to avoid negative quantities, the formula in this case should be written

$$\frac{b-a}{b+a}=\frac{\tan\frac{1}{2}(B-A)}{\tan\frac{1}{2}(B+A)}.$$

EXERCISE XII.

1. What do the formulas of § 36 become when one of the angles is a right angle?

2. Prove by means of the Law of Sines that the bisector of an angle of a triangle divides the opposite side into parts proportional to the adjacent sides.

3. What does Formula [26] become when $A=90°$? when $A=0°$? when $A=180°$? What does the triangle become in each of these cases?

NOTE. The case where $A=90°$ explains why the theorem of § 37 is sometimes termed the *Generalized Theorem of Pythagoras.*

4. Prove (Figs. 30 and 31) that whether the angle B is acute or obtuse $c=a\cos B+b\cos A$. What are the two symmetrical formulas obtained by changing the letters? What does the formula become when $B=90°$?

5. From the three following equations (found in the last exercise) prove the theorem of § 37:

$$c=a\cos B+b\cos A,$$
$$b=a\cos C+c\cos A,$$
$$a=b\cos C+c\cos B.$$

HINT. Multiply the first equation by c, the second by b, the third by a; then from the first subtract the *sum* of the second and third.

6. In Formula [27] what is the maximum value of $\frac{1}{2}(A-B)$? of $\frac{1}{2}(A+B)$?

7. Find the form to which Formula [27] reduces, and describe the nature of the triangle, when

(i.) $C=90°$; (ii.) $A-B=90°$, and $B=C$.

§ 39. The Given Parts of an Oblique Triangle.

The formulas established in §§ 36–38, together with the equation $A+B+C=180°$, are sufficient for solving every case of an oblique triangle. The three parts that determine an oblique triangle may be:

I. One side and two angles;
II. Two sides and the angle opposite to one of these sides;
III. Two sides and the included angle;
IV. The three sides.

In all cases let A, B, C denote the angles, a, b, c the sides opposite these angles respectively.

§ 40. Case I.

Given one side a, and two angles A and B; find the remaining parts C, b, and c.

1. $C=180°-(A+B)$.

2. $$\frac{b}{a}=\frac{\sin B}{\sin A}; \quad \therefore b=\frac{a\sin B}{\sin A}=\frac{a}{\sin A}\times\sin B.$$

3. $$\frac{c}{a}=\frac{\sin C}{\sin A}; \quad \therefore c=\frac{a\sin C}{\sin A}=\frac{a}{\sin A}\times\sin C.$$

Example. $a=24.31$, $A=45°\,18'$, $B=22°\,11'$.

The work may be arranged as follows:

$a=24.31$	$\log a=1.38578$	$=1.38578$
$A=45°\,18'$	$\operatorname{colog}\sin A=0.14825$	$=0.14825$
$B=\underline{22°\,11'}$	$\log\sin B=\underline{9.57700}$	$\log\sin C=\underline{9.96556}$
$A+B=67°\,29'$	$\log b=1.11103$	$\log c=1.49959$
$C=112°\,31'$	$b=12.913$	$c=31.593$

EXERCISE XIII.

1. Given $a = 500$, $A = 10^\circ 12'$, $B = 46^\circ 36'$;
find $C = 123^\circ 12'$, $b = 2051.48$, $c = 2362.61$.
2. Given $a = 795$, $A = 79^\circ 59'$, $B = 44^\circ 41'$;
find $C = 55^\circ 20'$, $b = 567.688$, $c = 663.986$.
3. Given $a = 804$, $A = 99^\circ 55'$, $B = 45^\circ 1'$;
find $C = 35^\circ 4'$, $b = 577.313$, $c = 468.933$.
4. Given $a = 820$, $A = 12^\circ 49'$, $B = 141^\circ 59'$;
find $C = 25^\circ 12'$, $b = 2276.63$, $c = 1573.89$.
5. Given $c = 1005$, $A = 78^\circ 19'$, $B = 54^\circ 27'$;
find $C = 47^\circ 14'$, $a = 1340.6$, $b = 1113.8$.
6. Given $b = 13.57$, $B = 13^\circ 57'$, $C = 57^\circ 13'$;
find $A = 108^\circ 50'$, $a = 53.276$, $c = 47.324$.
7. Given $a = 6412$, $A = 70^\circ 55'$, $C = 52^\circ 9'$;
find $B = 56^\circ 56'$, $b = 5685.9$, $c = 5357.5$.
8. Given $b = 999$, $A = 37^\circ 58'$, $C = 65^\circ 2'$;
find $B = 77^\circ$, $a = 630.77$, $c = 929.48$.

9. In order to determine the distance of a hostile fort A from a place B, a line BC and the angles ABC and BCA were measured, and found to be 322.55 yards, $60^\circ 34'$, and $56^\circ 10'$, respectively. Find the distance AB.

10. In making a survey by triangulation, the angles B and C of a triangle ABC were found to be $50^\circ 30'$ and $122^\circ 9'$, respectively, and the length BC is known to be 9 miles. Find AB and AC.

11. Two observers 5 miles apart on a plain, and facing each other, find that the angles of elevation of a balloon in the same vertical plane with themselves are 55° and 58°, respectively. Find the distance from the balloon to each observer, and also the height of the balloon above the plain.

12. In a parallelogram given a diagonal d and the angles x and y which this diagonal makes with the sides. Find the sides. Compute the results when $d = 11.237$, $x = 19^\circ 1'$, and $y = 42^\circ 54$.

13. A lighthouse was observed from a ship to bear N. 34° E.; after sailing due south 3 miles, it bore N. 23° E. Find the distance from the lighthouse to the ship in both positions.

NOTE. The phrase *to bear N.* 34° *E.* means that the line of sight to the lighthouse is in the north-east quarter of the horizon, and makes, with a line due north, an angle of 34°.

14. In a trapezoid given the parallel sides a and b, and the angles x and y at the ends of one of the parallel sides. Find the non-parallel sides. Compute the results when $a = 15$, $b = 7$, $x = 70°$, $y = 40°$.

Solve the following examples without using logarithms:

15. Given $b = 7.07107$, $A = 30°$, $C = 105°$; find a and c.

16. Given $c = 9.562$, $A = 45°$, $B = 60°$; find a and b.

17. The base of a triangle is 600 feet, and the angles at the base are 30° and 120°. Find the other sides and the altitude.

18. Two angles of a triangle are, the one 20°, the other 40°. Find the ratio of the opposite sides.

19. The angles of a triangle are as 5 : 10 : 21, and the side opposite the smallest angle is equal to 3. Find the other sides.

20. Given one side of a triangle equal to 27, the adjacent angles equal each to 30°. Find the radius of the circumscribed circle. (See § 36, Remark.)

§ 41. CASE II.

Given two sides a and b, and the angle A opposite to the side a; find the remaining parts B, C, c.

This case, like the preceding case, is solved by means of the Law of Sines.

Since $\frac{\sin B}{\sin A} = \frac{b}{a}$, therefore $\sin B = \frac{b \sin A}{a}$;

$$C = 180° - (A + B).$$

And since $\frac{c}{a} = \frac{\sin C}{\sin A}$, therefore $c = \frac{a \sin C}{\sin A}$.

When an angle is determined by its sine it admits of two values, which are supplements of each other (§ 28); hence, either value of B may be taken unless excluded by the conditions of the problem.

If $a > b$, then by Geometry $A > B$, and B must be acute whatever be the value of A; for a triangle can have only one obtuse angle. Hence, there is *one, and only one, triangle* that will satisfy the given conditions.

If $a = b$, then by Geometry $A = B$; both A and B must be acute, and *the required triangle is isosceles.*

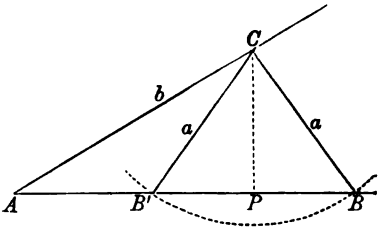

Fig. 33.

If $a < b$, then by Geometry $A < B$, and A must be acute in order that the triangle may be possible. If A is acute, it is evident from Fig. 33, where $\angle BAC = A$, $AC = b$, $CB = CB' = a$, that *the two triangles ACB and ACB' will satisfy the given conditions,* provided a is greater than the perpendicular CP; that is, provided a is greater than $b \sin A$ (§ 10). The angles ABC and $AB'C$ are supplementary (since $\angle ABC = \angle BB'C$); they are in fact the supplementary angles obtained from the formula

$$\sin B = \frac{b \sin A}{a}.$$

If, however, $a = b \sin A = CP$ (Fig. 33), then $\sin B = 1$, $B = 90°$, and *the triangle required is a right triangle.*

If $a < b \sin A$, that is, $< CP$, then $\sin B > 1$, and *the triangle is impossible.*

These results, for convenience, may be thus stated:

If $a > b$, or $a = b$, or if $a = b \sin A$, *One* solution.

If $a < b$, but $> b \sin A$, and $A < 90°$, *Two* solutions.

If $a < b$ and $A > 90°$, or if $a < b \sin A$ and $A < 90°$, *No* solution.

The number of solutions can often be determined by inspection. If there is any doubt, it may be removed by computing the value of $b \sin A$.

Or we may proceed to compute $\log \sin B$. If $\log \sin B = 0$, the triangle required is a right triangle. If $\log \sin B > 0$, the triangle is impossible. If $\log \sin B < 0$, there is *one* solution when $a > b$; there are *two* solutions when $a < b$.

When there are two solutions, let B', C', c', denote the unknown parts of the second triangle; then,

$$B' = 180° - B, \quad C' = 180° - (A + B') = B - A,$$

$$c' = \frac{a \sin C'}{\sin A}.$$

EXAMPLES.

1. Given $a = 16$, $b = 20$, $A = 106°$; find the remaining parts.

In this case $a < b$, and $A > 90°$; therefore the triangle is impossible.

2. Given $a = 36$, $b = 80$, $A = 30°$; find the remaining parts.

Here we have $b \sin A = 80 \times \frac{1}{2} = 40$; so that $a < b \sin A$, and the triangle is impossible.

3. Given $a = 72630$, $b = 117480$, $A = 80° 0' 50''$; find B, C, c.

$a = 72630$	$\text{colog}\, a = 5.13888$	Here $\log \sin B > 0$.
$b = 117480$	$\log b = 5.06996$	$\therefore$ *no* solution.
$A = 80° 0' 50''$	$\log \sin A = 9.99337$	
	$\log \sin B = 0.20221$	

4. Given $a = 13.2$, $b = 15.7$, $A = 57° 13' 15.3''$; find B, C, c.

$a = 13.2$	$\text{colog}\, a = 8.87943$	$c = b \cos A$
$b = 15.7$	$\log b = 1.19590$	$\log b = 1.19590$
$A = 57° 13' 15.3''$	$\log \sin A = 9.92467$	$\log \cos A = 9.73352$
Here $\log \sin B = 0$,	$\log \sin B = 0.00000$	$\log c = 0.92942$
$\therefore$ a *right* triangle.	$B = 90°$	$c = 8.5$
	$\therefore C = 32° 46' 44.7''$	

5. Given $a = 767$, $b = 242$, $A = 36° 53' 2''$; find B, C, c.

$a = 767$	$\operatorname{colog} a = 7.11520$	$\log a = 2.88480$
$b = 242$	$\log b = 2.38382$	$\log \sin C = 9.86970$
$A = 36° 53' 2''$	$\log \sin A = 9.77830$	$\operatorname{colog} \sin A = 0.22170$
Here $a > b$,	$\log \sin B = 9.27732$	$\log c = 2.97620$
and $\log \sin B < 0$.	$B = 10° 54' 58''$	$c = 946.675$
∴ *one* solution.	$\therefore C = 132° 12' 0''$	

6. Given $a = 177.01$, $b = 216.45$, $A = 35° 36' 20''$; find the other parts.

$a = 177.01$	$\operatorname{colog} a = 7.75200$	$\log a = 2.24800$	2.24800
$b = 216.45$	$\log b = 2.33536$	$\operatorname{colog} \sin A = 0.23493$	0.23493
$A = 35° 36' 20''$	$\log \sin A = 9.76507$	$\log \sin C = 9.99462$	9.23034
Here $a < b$,	$\log \sin B = 9.85243$	$\log c = 2.47755$	1.71327
and $\log \sin B < 0$.	$B = 45° 23' 28''$	$c = 300.29$ or	51.674
∴ *two* solutions.	or $134° 36' 32''$		
	$\therefore C = 99° 0' 12''$		
	or $9° 47' 8''$		

EXERCISE XIV.

1. Determine the number of solutions in each of the following cases:

(i.) $a = 80$, $b = 100$, $A = 30°$.
(ii.) $a = 50$, $b = 100$, $A = 30°$.
(iii.) $a = 40$, $b = 100$, $A = 30°$.
(iv.) $a = 13.4$, $b = 11.46$, $A = 77° 20'$.
(v.) $a = 70$, $b = 75$, $A = 60°$.
(vi.) $a = 134.16$, $b = 84.54$, $B = 52° 9' 11''$.

2. Given $a = 840$, $b = 485$, $A = 21° 31'$;
find $B = 12° 13' 34''$, $C = 146° 15' 26''$, $c = 1272.18$.

3. Given $a = 9.399$, $b = 9.197$, $A = 120° 35'$;
find $B = 57° 23' 40''$, $C = 2° 1' 20'$, $c = 0.38525$.

4. Given $a = 91.06$, $b = 77.04$, $A = 51° 9' 6''$;
find $B = 41° 13'$, $C = 87° 37' 54''$, $c = 116.82$.

5. Given $a = 55.55$, $b = 66.66$, $B = 77° 44' 40''$;
find $A = 54° 31' 13''$, $C = 47° 44' 7''$, $c = 50.481$.

6. Given $a = 309$, $b = 360$, $A = 21^\circ 14' 25''$;
find $B = 24^\circ 57' 54''$, $C = 133^\circ 47' 41''$, $c = 615.67$,
$B' = 155^\circ 2' 6''$, $C' = 3^\circ 43' 29''$, $c' = 55.41$.

7. Given $a = 8.716$, $b = 9.787$, $A = 38^\circ 14' 12''$;
find $B = 44^\circ 1' 28''$, $C = 97^\circ 44' 20''$, $c = 13.954$,
$B' = 135^\circ 58' 32''$, $C' = 5^\circ 47' 16''$, $c' = 1.4203$.

8. Given $a = 4.4$, $b = 5.21$, $A = 57^\circ 37' 17''$;
find $B = 90^\circ$, $C = 32^\circ 22' 43''$, $c = 2.79$.

9. Given $a = 34$, $b = 22$, $B = 30^\circ 20'$;
find $A = 51^\circ 18' 27''$, $C = 98^\circ 21' 33''$, $c = 43.098$,
$A' = 128^\circ 41' 33''$, $C' = 20^\circ 58' 27''$, $c' = 15.593$.

10. Given $b = 19$, $c = 18$, $C = 15^\circ 49'$;
find $B = 16^\circ 43' 13''$, $A = 147^\circ 27' 47''$, $a = 35.519$,
$B' = 163^\circ 16' 47''$, $A' = 0^\circ 54' 13''$, $a' = 1.0415$,

11. Given $a = 75$, $b = 29$, $B = 16^\circ 15' 36''$; find the difference between the areas of the two corresponding triangles.

12. Given in a parallelogram the side a, a diagonal d, and the angle A made by the two diagonals; find the other diagonal.

Special case: $a = 35$, $d = 63$, $A = 21^\circ 36' 30''$.

§ 42. CASE III.

Given two sides a and b and the included angle C; find the remaining parts A, B, and c.

SOLUTION I. The angles A and B may both be found by means of Formula [27], § 38, which may be written

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} \times \tan \tfrac{1}{2}(A + B).$$

Since $\frac{1}{2}(A + B) = \frac{1}{2}(180^\circ - C)$, the value of $\frac{1}{2}(A + B)$ is known; so that this equation enables us to find the value of $\frac{1}{2}(A - B)$. We then have

$$\tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B) = A,$$

and $$\tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B) = B.$$

After A and B are known, the side c may be found by the Law of Sines, which gives its value in two ways, as follows:

$$c = \frac{a \sin C}{\sin A}, \quad \text{or} \quad c = \frac{b \sin C}{\sin B}.$$

SOLUTION II. The third side c may be found directly from the equation (§ 37)

$$c = \sqrt{a^2 + b^2 - 2ab \cos C};$$

and then, by the Law of Sines, the following equations for computing the values of the angles A and B are obtained:

$$\sin A = a \times \frac{\sin C}{c}, \quad \sin B = b \times \frac{\sin C}{c}.$$

SOLUTION III. If, in the triangle ABC (Fig. 34), BD is drawn perpendicular to the side AC, then

$$\tan A = \frac{BD}{AD} = \frac{BD}{AC - DC}.$$

Now $BD = a \sin C$ (§ 10),

and $DC = a \cos C$.

$$\therefore \tan A = \frac{a \sin C}{b - a \cos C}.$$

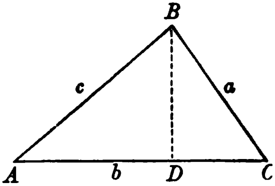

Fig. 34.

By merely changing the letters,

$$\tan B = \frac{b \sin C}{a - b \cos C}.$$

It is not necessary, however, to use both formulas. When one angle, as A, has been found, the other, B, may be found from the relation $A + B + C = 180°$.

When the angles are known, the third side is found by the Law of Sines, as in Solution I.

NOTE. When all three unknown parts are required, Solution I. is the most convenient in practice. When only the third side c is desired, Solution II. may be used to advantage, provided the values of a^2 and b^2 can be readily obtained without the aid of logarithms. But Solutions II. and III. are not adapted to logarithmic work.

EXAMPLES.

1. Given $a = 748$, $b = 375$, $C = 63° 35' 30''$; find A, B, and c.

$a + b = 1123$		
$a - b = 373$	$\log(a-b) = 2.57171$	$\log b = 2.57403$
$(A+B) = 116° 24' 30''$	$\operatorname{colog}(a+b) = 6.94961$	$\log \sin C = 9.95214$
$\frac{1}{2}(A+B) = 58° 12' 15''$	$\log \tan \frac{1}{2}(A+B) = 0.20766$	$\operatorname{colog} \sin B = 0.30073$
$\frac{1}{2}(A-B) = 28° 10' 52''$	$\log \tan \frac{1}{2}(A-B) = 9.72898$	$\log c = 2.82690$
$A = 86° 23' 7''$	$\frac{1}{2}(A-B) = 28° 10' 52''$	$c = 671.27$
$B = 30° 1' 23''$		

NOTE. In the above Example we use the angle B in finding the side c, rather than the angle A, because A is near 90°, and therefore its sine should be avoided.

2. Given $a = 4$, $c = 6$, $B = 60°$; find the third side b.

Here Solution II. may be used to advantage. We have

$$b = \sqrt{a^2 + c^2 - 2ac \cos B} = \sqrt{16 + 36 - 24} = \sqrt{28};$$

$$\log 28 = 1.44716, \quad \log \sqrt{28} = 0.72358, \quad \sqrt{28} = 5.2915;$$

that is, $b = 5.2915$.

EXERCISE XV.

1. Given $a = 77.99$, $b = 83.39$, $C = 72° 15'$; [§ 38, N.
find $A = 51° 15'$, $B = 56° 30'$, $c = 95.24$.
2. Given $b = 872.5$, $c = 632.7$, $A = 80°$;
find $B = 60° 45'$, $C = 39° 15'$, $a = 984.83$.
3. Given $a = 17$, $b = 12$, $C = 59° 17'$;
find $A = 77° 12' 53''$, $B = 43° 30' 7''$, $c = 14.987$.
4. Given $b = \sqrt{5}$, $c = \sqrt{3}$, $A = 35° 53'$;
find $B = 93° 28' 36''$, $C = 50° 38' 24''$, $a = 1.313$.
5. Given $a = 0.917$, $b = 0.312$, $C = 33° 7' 9''$;
find $A = 132° 18' 27''$, $B = 14° 34' 24''$, $c = 0.67748$.
6. Given $a = 13.715$, $c = 11.214$, $B = 15° 22' 36''$;
find $A = 118° 55' 49''$, $C = 45° 41' 35''$, $b = 4.1554$.
7. Given $b = 3000.9$, $c = 1587.2$, $A = 86° 4' 4''$;
find $B = 65° 13' 51''$, $C = 28° 42' 5''$, $a = 3297.2$.

8. Given $a = 4527$, $b = 3465$, $C = 66° 6' 27''$;
find $A = 68° 29' 15''$, $B = 45° 24' 18''$, $c = 4449$.

9. Given $a = 55.14$, $b = 33.09$, $C = 30° 24'$;
find $A = 117° 24' 33''$, $B = 32° 11' 27''$, $c = 31.431$.

10. Given $a = 47.99$, $b = 33.14$, $C = 175° 19' 10''$;
find $A = 2° 46' 8''$, $B = 1° 54' 42''$, $c = 81.066$.

11. If two sides of a triangle are each equal to 6, and the included angle is 60°, find the third side.

12. If two sides of a triangle are each equal to 6, and the included angle is 120°, find the third side.

13. Apply Solution I. to the case in which $a = b$ or the triangle is isosceles.

14. If two sides of a triangle are 10 and 11, and the included angle is 50°, find the third side.

15. If two sides of a triangle are 43.301 and 25, and the included angle is 30°, find the third side.

16. In order to find the distance between two objects A and B separated by a swamp, a station C was chosen, and the distances $CA = 3825$ yards, $CB = 3475.6$ yards, together with the angle $ACB = 62° 31'$, were measured. Find the distance from A to B.

17. Two inaccessible objects A and B are each viewed from two stations C and D 562 yards apart. The angle ACB is $62° 12'$, BCD $41° 8'$, ADB $60° 49'$, and ADC $34° 51'$; required the distance AB.

18. Two trains start at the same time from the same station, and move along straight tracks that form an angle of 30°, one train at the rate of 30 miles an hour, the other at the rate of 40 miles an hour. How far apart are the trains at the end of half an hour?

19. In a parallelogram given the two diagonals 5 and 6, and the angle that they form $49° 18'$. Find the sides.

20. In a triangle one angle $= 139° 54'$, and the sides forming the angle have the ratio $5 : 9$. Find the other two angles.

§ 43. CASE IV.

Given the three sides a, b, c; find the angles A, B, C.

The angles may be found directly from the formulas established in § 37. Thus, from the formula

$$a^2 = b^2 + c^2 - 2bc\cos A$$

we have $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

From this equation formulas adapted to logarithmic work are deduced as follows:

For the sake of brevity, let $a + b + c = 2s$; then $b + c - a = 2(s - a)$, $a - b + c = 2(s - b)$, and $a + b - c = 2(s - c)$.

Then the value of $1 - \cos A$ is

$$1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc - b^2 - c^2 + a^2}{2bc} = \frac{a^2 - (b - c)^2}{2bc}$$
$$= \frac{(a + b - c)(a - b + c)}{2bc} = \frac{2(s - b)(s - c)}{bc};$$

and the value of $1 + \cos A$ is

$$1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc + b^2 + c^2 - a^2}{2bc} = \frac{(b + c)^2 - a^2}{2bc}$$
$$= \frac{(b + c + a)(b + c - a)}{2bc} = \frac{2s(s - a)}{bc}.$$

But from Formulas [16] and [17], § 34, it follows that

$$1 - \cos A = 2\sin^2 \tfrac{1}{2} A, \text{ and } 1 + \cos A = 2\cos^2 \tfrac{1}{2} A.$$

$$\therefore\ 2\sin^2 \tfrac{1}{2} A = \frac{2(s - b)(s - c)}{bc}, \text{ and } 2\cos^2 \tfrac{1}{2} A = \frac{2s(s - a)}{bc},$$

whence $$\sin \tfrac{1}{2} A = \sqrt{\frac{(s - b)(s - c)}{bc}}, \qquad [28]$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s - a)}{bc}}, \qquad [29]$$

and therefore $$\tan \tfrac{1}{2} A = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}. \qquad [30]$$

By merely changing the letters,

$$\sin \tfrac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{ac}}, \quad \sin \tfrac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{ab}}.$$

$$\cos \tfrac{1}{2} B = \sqrt{\frac{s(s-b)}{ac}}, \quad \cos \tfrac{1}{2} C = \sqrt{\frac{s(s-c)}{ab}}.$$

$$\tan \tfrac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \quad \tan \tfrac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

There is then a choice of three different formulas for finding the value of each angle. If half the angle is very near 0°, the formula for the cosine will not give a very accurate result, because the cosines of angles near 0° differ little in value; and the same holds true of the formula for the sine when half the angle is very near 90°. Hence, in the first case the formula for the sine, in the second that for the cosine, should be used.

But, in general, the formulas for the tangent are to be preferred.

It is not necessary to compute by the formulas more than two angles; for the third may then be found from the equation

$$A + B + C = 180^\circ.$$

There is this advantage, however, in computing all three angles by the formulas, that we may then use the sum of the angles as a test of the accuracy of the results.

In case it is desired to compute all the angles, the formulas for the tangent may be put in a more convenient form.

The value of $\tan \frac{1}{2} A$ may be written

$$\sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)^2}} \text{ or } \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Hence, if we put

$$\sqrt{\frac{(\mathbf{s}-\mathbf{a})(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})}{\mathbf{s}}} = \mathbf{r}, \qquad [31]$$

we have

$$\tan \tfrac{1}{2} \mathbf{A} = \frac{\mathbf{r}}{\mathbf{s}-\mathbf{a}}. \qquad [32]$$

In like manner,

$$\tan \tfrac{1}{2} B = \frac{r}{s-b}, \qquad \tan \tfrac{1}{2} C = \frac{r}{s-c}.$$

EXAMPLES.

1. Given $a = 3.41$, $b = 2.60$, $c = 1.58$; find the angles.

Using Formula [30], and the corresponding formula for $\tan \frac{1}{2} B$, we may arrange the work as follows:

$a = 3.41$	$\operatorname{colog} s = 9.42079$	$\operatorname{colog} s = 9.42079 - 10$
$b = 2.60$	$\operatorname{colog}(s-a) = 0.41454$	$\log(s-a) = 9.58546 - 10$
$c = 1.58$	$\log(s-b) = 0.07737$	$\operatorname{colog}(s-b) = 9.92263 - 10$
$2s = 7.59$	$\log(s-c) = \underline{0.34537}$	$\log(s-c) = \underline{0.34537}$
$s = 3.795$	$2\overline{)0.25807}$	$2\overline{)19.27425 - 20}$
$s-a = 0.385$	$\log \tan \frac{1}{2} A = 0.12903$	$\log \tan \frac{1}{2} B = 9.63713 - 10$
$s-b = 1.195$	$\frac{1}{2} A = 53° 23' 20''$	$\frac{1}{2} B = 23° 26' 37''$
$s-c = 2.215$	$A = 106° 46' 40''$	$B = 46° 53' 14''$

$\therefore A + B = 153° 39' 54''$, and $C = 26° 20' 6''$.

2. Solve Example 1 by finding all three angles by the use of Formulas [31] and [32].

Here the work may be compactly arranged as follows, if we find $\log \tan \frac{1}{2} A$, etc., by *subtracting* $\log(s-a)$, etc., from $\log r$ instead of adding the cologarithm:

$a = 3.41$	$\log(s-a) = 9.58546$	$\log \tan \frac{1}{2} A = 10.12903$
$b = 2.60$	$\log(s-b) = 0.07737$	$\log \tan \frac{1}{2} B = 9.63713$
$c = \underline{1.58}$	$\log(s-c) = 0.34537$	$\log \tan \frac{1}{2} C = \underline{9.36912}$
$2s = \underline{7.59}$	$\operatorname{colog} s = \underline{9.42079}$	$\frac{1}{2} A = 53° 23' 20''$
$s = 3.795$	$\log r^2 = 9.42899$	$\frac{1}{2} B = 23° 26' 37''$
$s-a = 0.385$	$\log r = 9.71450$	$\frac{1}{2} C = \underline{13° 10' 3''}$
$s-b = 1.195$		$A = 106° 46' 40''$
$s-c = \underline{2.215}$		$B = 46° 53' 14''$
$2s = 7.590$ (proof).		$C = \underline{26° 20' 6''}$
		Proof, $A + B + C = 180° 0' 0''$

NOTE. Even if no mistakes are made in the work the sum of the three angles found as above may differ very slightly from 180° in consequence of the fact that logarithmic computation is at best only a method of close approximation. When a difference of this kind exists it should be divided among the angles according to the probable amount of error for each angle.

Exercise XVI.

Solve the following triangles, taking the three sides as the given parts:

	a	b	c	A	B	C
1	51	65	20	38° 52′ 48″	126° 52′ 12″	14° 15′
2	78	101	29	32° 10′ 54″	136° 23′ 50″	11° 25′ 16″
3	111	145	40	27° 20′ 32″	143° 7′ 48″	9° 31′ 40″
4	21	26	31	42° 6′ 13″	56° 6′ 36″	81° 47′ 11″
5	19	34	49	16° 25′ 36″	30° 24′	133° 10′ 24″
6	43	50	57	46° 49′ 35″	57° 59′ 44″	75° 10′ 41″
7	37	58	79	26° 0′ 29″	43° 25′ 20″	110° 34′ 11″
8	73	82	91	49° 34′ 58″	58° 46′ 58″	71° 38′ 4″
9	14.493	55.4363	66.9129	8° 20′	33° 40′	138°
10	$\sqrt{5}$	$\sqrt{6}$	$\sqrt{7}$	51° 53′ 12″	59° 31′ 48″	68° 35′

11. Given $a = 6$, $b = 8$, $c = 10$; find the angles.
12. Given $a = 6$, $b = 6$, $c = 10$; find the angles.
13. Given $a = 6$, $b = 6$, $c = 6$; find the angles.
14. Given $a = 6$, $b = 5$, $c = 12$; find the angles.
15. Given $a = 2$, $b = \sqrt{6}$, $c = \sqrt{3} - 1$; find the angles.
16. Given $a = 2$, $b = \sqrt{6}$, $c = \sqrt{3} + 1$; find the angles.
17. The distances between three cities A, B, and C are as follows: $AB = 165$ miles, $AC = 72$ miles, and $BC = 185$ miles. B is due east from A. In what direction is C from A? What two answers are admissible?
18. Under what visual angle is an object 7 feet long seen by an observer whose eye is 5 feet from one end of the object and 8 feet from the other end?
19. When Formula [28] is used for finding the value of an angle, why does the ambiguity that occurs in Case II. not exist?
20. If the sides of a triangle are 3, 4, and 6, find the sine of the largest angle.
21. Of three towns A, B, and C, A is 200 miles from B and 184 miles from C, B is 150 miles due north from C; how far is A north of C?

§ 44. AREA OF A TRIANGLE.

If F denote the area of the triangle ABC (Fig. 30 or 31, page 50), then, by Geometry,

$$F = \tfrac{1}{2} c \times CD.$$

By § 10, $$CD = a \sin B.$$

Therefore, $$\mathbf{F = \tfrac{1}{2} ac \sin B.} \qquad [33]$$

And, in like manner,

$$F = \tfrac{1}{2} ab \sin C \quad \text{and} \quad F = \tfrac{1}{2} bc \sin A.$$

That is: *The area of a triangle is equal to half the product of two sides and the sine of the included angle.*

By Formula [33] the area of a triangle may be found directly when two sides and the included angle are given; in the other cases the formula may be used when these parts have been computed.

When the three sides of a triangle are given, as in Case IV., a formula for its area may be found as follows:

By § 33, $\sin B = 2 \sin \tfrac{1}{2} B \times \cos \tfrac{1}{2} B.$

By substituting for $\sin \tfrac{1}{2} B$ and $\cos \tfrac{1}{2} B$ their values in terms of the sides given in § 43,

$$\sin B = \frac{2}{ac} \sqrt{s(s-a)(s-b)(s-c)}.$$

By substituting this value of $\sin B$ in [33],

$$\mathbf{F = \sqrt{s(s-a)(s-b)(s-c)}.} \qquad [34]$$

If R denote (as in § 36) the radius of the circumscribed circle, we have, from § 36,

$$\sin B = \frac{b}{2R}.$$

By substituting this value of $\sin B$ in [33],

$$\mathbf{F = \frac{abc}{4R}.} \qquad [35]$$

If r denote the radius of the inscribed circle, and we divide the triangle into three triangles by lines from the centre of this circle to the vertices, the altitude of each of the three triangles is equal to r. Therefore,

$$\mathbf{F} = \tfrac{1}{2}\mathbf{r}(\mathbf{a}+\mathbf{b}+\mathbf{c}) = \mathbf{rs}. \qquad [36]$$

By substituting in this formula the value of F given in [34],

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}};$$

whence r, in [31] § 43, is equal to the radius of the inscribed circle.

EXERCISE XVII.

Find the area:

1. Given $a = 4474.5$, $b = 2164.5$, $C = 116° 30' 20''$.
2. Given $b = 21.66$, $c = 36.94$, $A = 66° 4' 19''$.
3. Given $a = 510$, $c = 173$, $B = 162° 30' 28''$.
4. Given $a = 408$, $b = 41$, $c = 401$.
5. Given $a = 40$, $b = 13$, $c = 37$.
6. Given $a = 624$, $b = 205$, $c = 445$.
7. Given $b = 149$, $A = 70° 42' 30''$, $B = 39° 18' 28''$.
8. Given $a = 215.9$, $c = 307.7$, $A = 25° 9' 31''$.
9. Given $b = 8$, $c = 5$, $A = 60°$.
10. Given $a = 7$, $c = 3$, $A = 60°$.
11. Given $a = 60$, $B = 40° 35' 12''$, area $= 12$; find the radius of the inscribed circle.
12. Obtain a formula for the area of a parallelogram in terms of two adjacent sides and the included angle.
13. Obtain a formula for the area of an isosceles trapezoid in terms of the two parallel sides and an acute angle.
14. Two sides and included angle of a triangle are 2416, 1712, and 30°; and two sides and included angle of another triangle are 1948, 2848, and 150°; find the sum of their areas.
15. The base of an isosceles triangle is 20, and its area is $100 \div \sqrt{3}$; find its angles.

EXERCISE XVIII.

1. From a ship sailing down the English Channel the Eddystone was observed to bear N. 33° 45′ W.; and after the ship had sailed 18 miles S. 67° 30′ W. it bore N. 11° 15′ E. Find its distance from each position of the ship.

2. Two objects, *A* and *B*, were observed from a ship to be at the same instant in a line bearing N. 15° E. The ship then sailed north-west 5 miles, when it was found that *A* bore due east and *B* bore north-east. Find the distance from *A* to *B*.

3. A castle and a monument stand on the same horizontal plane. The angles of depression of the top and the bottom of the monument viewed from the top of the castle are 40° and 80°; the height of the castle is 140 feet. Find the height of the monument.

4. If the sun's altitude is 60°, what angle must a stick make with the horizon in order that its shadow in a horizontal plane may be the longest possible?

5. If the sun's altitude is 30°, find the length of the longest shadow cast on a horizontal plane by a stick 10 feet in length.

6. In a circle with the radius 3 find the area of the part comprised between parallel chords whose lengths are 4 and 5 (Two solutions.)

7. *A* and *B*, two inaccessible objects in the same horizontal plane, are observed from a balloon at *C* and from a point *D* directly under the balloon, and in the same horizontal plane with *A* and *B*. If $CD = 2000$ yards, $\angle ACD = 10° 15' 10''$, $\angle BCD = 6° 7' 20''$, $\angle ADB = 49° 34' 50''$, find *AB*.

8. *A* and *B* are two objects whose distance, on account of intervening obstacles, cannot be directly measured. At the summit *C* of a hill, whose height above the common horizontal plane of the objects is known to be 517.3 yards, $\angle ACB$ is found to be 15° 13′ 15″. The angles of elevation of *C* viewed from *A* and *B* are 21° 9′ 18″ and 23° 15′ 34″ respectively. Find the distance from *A* to *B*.

MISCELLANEOUS PROBLEMS.

[Selected by permission from "Problems in Plane Trigonometry," prepared by Prof. C. J. White, of Harvard College, and published by Charles W. Sever, Cambridge.]

1. The *angular distance* of any object from a horizontal plane, as observed at any point of that plane, is the angle which a line drawn from the object to the point of observation makes with the plane. If the object observed be situated above the horizontal plane (that is, if it is farther from the earth's centre than the plane is), its angular distance from the plane is called its *angle of elevation.* If the object be below the plane, its angular distance from the plane is called its *angle of depression.* These angles are evidently vertical angles.

If two objects are in the same horizontal plane with the point of observation, the angular distance of one object from the other is called its *bearing* from that object.

If two objects are not in the same horizontal plane with either each other or the point of observation, we may suppose vertical lines to be passed through the two objects, and to meet the horizontal plane of the point of observation in two points. The angular distance of these two points is the bearing of either of the objects from the other. It may also be called the *horizontal distance* of one object from the other.

NOTE. "Problems in Plane Trigonometry" can be obtained in pamphlet form of Charles W. Sever, Cambridge, Mass.

RIGHT TRIANGLES.

2. The angle of elevation of a tower is 48° 19′ 14″, and the distance of its base from the point of observation is 95 ft. Find the height of the tower, and the distance of its top from the point of observation.

3. From a mountain 1000 ft. high, the angle of depression of a ship is 77° 35′ 11″. Find the distance of the ship from the summit of the mountain.

4. A flag-staff 90 ft. high, on a horizontal plane, casts a shadow of 117 ft. Find the altitude of the sun.

5. When the moon is setting at any place, the angle at the moon subtended by the earth's radius passing through that place is 57′ 3″. If the earth's radius is 3956.2 miles, what is the moon's distance from the earth's centre?

6. The angle at the earth's centre subtended by the sun's radius is 16′ 2″, and the sun's distance is 92,400,000 miles. Find the sun's diameter in miles.

7. The latitude of Cambridge, Mass., is 42° 22′ 49″. What is the length of the radius of that parallel of latitude?

8. At what latitude is the circumference of the parallel half of that of the equator?

9. In a circle with a radius of 6.7 is inscribed a regular polygon of thirteen sides. Find the length of one of its sides.

10. A regular heptagon, one side of which is 5.73, is inscribed in a circle. Find the radius of the circle.

11. A tower 93.97 ft. high is situated on the bank of a river. The angle of depression of an object on the opposite bank is 25° 12′ 54″. Find the breadth of the river.

12. From a tower 58 ft. high the angles of depression of two objects situated in the same horizontal line with the base of the tower, and on the same side, are 30° 13′ 18″ and 45° 46′ 14″. Find the distance between these two objects.

13. Standing directly in front of one corner of a flat-roofed house, which is 150 ft. in length, I observe that the horizontal angle which the length subtends has for its cosine $\sqrt{\frac{1}{5}}$, and that the vertical angle subtended by its height has for its sine $\frac{3}{\sqrt{34}}$. What is the height of the house?

14. A regular pyramid, with a square base, has an edge 150 ft. in length, and the length of a side of its base is 200 ft. Find the inclination of the face of the pyramid to the base.

15. From one edge of a ditch 36 ft. wide, the angle of elevation of a wall on the opposite edge is 62° 39′ 10″. Find the length of a ladder which will reach from the point of observation to the top of the wall.

16. The top of a flag-staff has been broken off, and touches the ground at a distance of 15 ft. from the foot of the staff. The length of the broken part being 39 ft., find the whole length of the staff.

17. From a balloon, which is directly above one town, is observed the angle of depression of another town, 10° 14′ 9″. The towns being 8 miles apart, find the height of the balloon.

18. From the top of a mountain 3 miles high the angle of depression of the most distant object which is visible on the earth's surface is found to be 2° 13′ 50″. Find the diameter of the earth.

19. A ladder 40 ft. long reaches a window 33 ft. high, on one side of a street. Being turned over upon its foot, it reaches another window 21 ft. high, on the opposite side of the street. Find the width of the street.

20. The height of a house subtends a right angle at a window on the other side of the street; and the elevation of the top of the house, from the same point, is 60°. The street is 30 ft. wide. How high is the house?

21. A lighthouse 54 ft. high is situated on a rock. The elevation of the top of the lighthouse, as observed from a ship, is 4° 52′, and the elevation of the top of the rock is 4° 2′. Find the height of the rock, and its distance from the ship.

22. A man in a balloon observes the angle of depression of an object on the ground, bearing south, to be 35° 30′; the balloon drifts $2\frac{1}{2}$ miles east at the same height, when the angle of depression of the same object is 23° 14′. Find the height of the balloon.

23. A man standing south of a tower, on the same horizontal plane, observes its elevation to be 54° 16′: he goes east 100 yds., and then finds its elevation is 50° 8′. Find the height of the tower.

24. The elevation of a tower at a place A south of it is 30°; and at a place B, west of A, and at a distance of a from it, the the elevation is 18°. Show that the height of the tower is $\dfrac{a}{\sqrt{(2+2\sqrt{5})}}$; the tangent of 18° being $\dfrac{\sqrt{5}-1}{\sqrt{(10+2\sqrt{5})}}$.

25. A pole is fixed on the top of a mound, and the angles of elevation of the top and the bottom of the pole are 60° and 30°. Prove that the length of the pole is twice the height of the mound.

26. At a distance (a) from the foot of a tower, the angle of elevation (A) of the top of the tower is the complement of the angle of elevation of a flag-staff on top of it. Show that the length of the staff is $2a \cot 2A$.

27. *A line of true level* is a line every point of which is equally distant from the centre of the earth. A line drawn

tangent to a line of true level at any point is a line of *apparent* level. If at any point both these lines be drawn, and extended one mile, find the distance they are then apart.

28. In Problem 2, determine the effect upon the computed height of the tower, of an error in either the angle of elevation or the measured distance.

OBLIQUE TRIANGLES.

29. To determine the height of an inaccessible object situated on a horizontal plane, by observing its angles of elevation at two points in the same line with its base, and measuring the distance of these two points.

30. The angle of elevation of an inaccessible tower, situated on a horizontal plane, is 63° 26′; at a point 500 ft. farther from the base of the tower the elevation of its top is 32° 14′. Find the height of the tower.

31. A tower is situated on the bank of a river. From the opposite bank the angle of elevation of the tower is 60° 13′, and from a point 40 ft. more distant the elevation is 50° 19′. Find the breadth of the river.

32. A ship sailing north sees two lighthouses 8 miles apart, in a line due west; after an hour's sailing, one lighthouse bears S.W., and the other S.S.W. Find the ship's rate.

33. To determine the height of an accessible object situated on an inclined plane.

34. At a distance of 40 ft. from the foot of a tower on an inclined plane, the tower subtends an angle of 41° 19′; at a point 60 ft. farther away, the angle subtended by the tower is 23° 45′. Find the height of the tower.

35. A tower makes an angle of 113° 12′ with the inclined plane on which it stands; and at a distance of 89 ft. from its base, measured down the plane, the angle subtended by the tower is 23° 27′. Find the height of the tower.

52. Wishing to find the height of a summit A, I measure a horizontal base line CD, 440 yds. At C, the elevation of A is 37° 18′, and the horizontal angle between D and the summit is 76° 18′; at D, the horizontal angle between C and the summit is 67° 14′. Find the height.

53. A balloon is observed from two stations 3000 ft. apart. At the first station the horizontal angle of the balloon and the other station is 75° 25′, and the elevation of the balloon is 18°. The horizontal angle of the first station and the balloon, measured at the second station, is 64° 30′. Find the height of the balloon.

54. Two forces, one of 410 pounds, and the other of 320 pounds, make an angle of 51° 37′. Find the intensity and the direction of their resultant.

55. An unknown force, combined with one of 128 pounds, produces a resultant of 200 pounds, and this resultant makes an angle of 18° 24′ with the known force. Find the intensity and direction of the unknown force.

56. At two stations, the height of a kite subtends the same angle (A). The angle which the line joining one station and the kite subtends at the other station is B; and the distance between the two stations is a. Show that the height of the kite is $\frac{1}{2}a \sin A \sec B$.

57. Two towers on a horizontal plane are 120 ft. apart. A person standing successively at their bases observes that the angular elevation of one is double that of the other; but, when he is half-way between them, the elevations are complementary. Prove that the heights of the towers are 90 and 40 ft.

58. To find the distance of an inaccessible point C from either of two points A and B, having no instruments to measure angles. Prolong CA to a, and CB to b, and join AB, Ab, and Ba. Measure AB, 500; aA, 100; aB, 560; bB, 100; and Ab, 550.

59. Two inaccessible points A and B, are visible from D, but no other point can be found whence both are visible. Take some point C, whence A and D can be seen, and measure CD 200 ft.; ADC, 89°; ACD, 50° 30′. Then take some point E, whence D and B are visible, and measure DE, 200; BDE, 54° 30′; BED, 88° 30′. At D measure ADB, 72° 30′. Compute the distance AB.

60. To compute the horizontal distance between two inaccessible points A and B, when no point can be found whence both can be seen. Take two points C and D, distant 200 yds. so that A can be seen from C, and B from D. From C measure CF, 200 yds. to F, whence A can be seen; and from D measure DE, 200 yds. to E, whence B can be seen. Measure AFC, 83°; ACD, 53° 30′; ACF, 54° 31′; BDE, 54° 30′; BDC, 156° 25′; DEB, 88° 30′.

61. A column in the north temperate zone is east-southeast of an observer, and at noon the extremity of its shadow is northeast of him. The shadow is 80 ft. in length, and the elevation of the column, at the observer's station, is 45°. Find the height of the column.

62. From the top of a hill the angles of depression of two objects situated in the horizontal plane of the base of the hill are 45° and 30°; and the horizontal angle between the two objects is 30°. Show that the height of the hill equals the distance between the objects.

63. Wishing to know the breadth of a river from A to B, I take AC, 100 yds. in the prolongation of BA, and then take CD, 200 yds. at right angles to AC. The angle BDA is 37° 18′ 30″. Find AB.

64. The sum of the sides of a triangle is 100. The angle at A is double that of B, and the angle at B is double that at C. Determine the sides.

65. If $\sin^2 A + 5\cos^2 A = 3$, find A.

66. If $\sin^2 A = m\cos A - n$, find $\cos A$.

67. Given $\sin A = m\sin B$, and $\tan A = n\tan B$, find $\sin A$ and $\cos B$.

68. If $\tan^2 A + 4\sin^2 A = 6$, find A.

69. If $\sin A = \sin 2A$, find A.

70. If $\tan 2A = 3\tan A$, find A.

71. Prove that $\tan 50^\circ + \cot 50^\circ = 2\sec 10^\circ$.

72. Given a regular polygon of n sides, and calling one of them a, find expressions for the radii of the inscribed and the circumscribed circle in terms of n and a.

If P, H, D be the sides of a regular inscribed pentagon, hexagon, and decagon, prove $P^2 = H^2 + D^2$.

AREAS.

73. Obtain the formula for the area of a triangle, given two sides b, c, and the included angle A.

74. Obtain the formula for the area of a triangle, given two angles A and B, and included side c.

75. Obtain the formula for the area of a triangle, given the three sides.

76. If a is the side of an equilateral triangle, its area is $\frac{a^2\sqrt{3}}{4}$.

77. Two consecutive sides of a rectangle are 52.25 ch. and 38.24 ch. Find its area.

78. Two sides of a parallelogram are 59.8 ch. and 37.05 ch., and the included angle is 72° 10′. Find the area.

79. Two sides of a parallelogram are 15.36 ch. and 11.46 ch., and the included angle is 47° 30′. Find its area.

80. Two sides of a triangle are 12.38 ch. and 6.78 ch., and the inclined angle is 46° 24′. Find the area.

81. Two sides of a triangle are 18.37 ch. and 13.44 ch., and they form a right angle. Find the area.

82. Two angles of a triangle are 76° 54′ and 57° 33′ 12″, and the included side is 9 ch. Find the area.

83. Two sides of a triangle are 19.74 ch. and 17.34 ch. The first bears N. 82° 30′ W.; the second, S. 24° 15′ E. Find the area.

84. The three sides of a triangle are 49 ch., 50.25 ch., and 25.69 ch. Find the area.

85. The three sides of a triangle are 10.64 ch., 12.28 ch., and 9 ch. Find the area.

86. The sides of a triangular field, of which the area is 14 acres, are in the ratio of 3, 5, 7. Find the sides.

87. In the quadrilateral $ABCD$ we have AB, 17.22 ch.; AD, 7.45 ch.; CD, 14.10 ch.; BC, 5.25 ch.; and the diagonal AC, 15.04 ch. Required the area.

88. The diagonals of a quadrilateral are a and b, and they intersect at an angle D. The area of the quadrilateral is $\frac{1}{2}ab\sin D$.

89. The diagonals of a quadrilateral are 34 and 56, intersecting at an angle of 67°. Find the area.

90. The diagonals of a quadrilateral are 75 and 49, intersecting at an angle of 42°. Find the area.

91. The area of a regular polygon of n sides, of which one is a, is $\frac{na^2}{4}\cot\frac{180^\circ}{n}$.

92. One side of a regular pentagon is 25. Find the area.

93. One side of a regular hexagon is 32. Find the area.

94. One side of a regular decagon is 46. Find the area.

95. Find the area of a circle whose circumference is 74 ft.

96. Find the area of a circle whose radius is 125 ft.

97. In a circle with a diameter of 125 ft. find the area of a sector with an arc of 22°.

98. In a circle with a radius of 44 ft. find the area of a sector with an arc of 25°.

99. In a circle with a diameter of 50 ft. find the area of a segment with an arc of 280°.

100. Find the area of a segment (less than a semicircle), of which the chord is 20, and the distance of the chord from the middle point of the smaller arc is 2.

101. If r is the radius of a circle, the area of a regular circumscribed polygon of n sides is $nr^2 \tan \frac{180°}{n}$.

The area of a regular inscribed polygon is $\frac{n}{2} r^2 \sin \frac{360°}{n}$.

102. If a is a side of a regular polygon of n sides, the area of the inscribed circle is $\frac{\pi a^2}{4} \cot^2 \frac{180°}{n}$.

The area of the circumscribed circle is $\frac{\pi a^2}{4} \csc^2 \frac{180°}{n}$.

103. The area of a regular polygon inscribed in a circle is to that of the circumscribed polygon of the same number of sides as 3 to 4. Find the number of sides.

104. The area of a regular polygon inscribed in a circle is a geometric mean between the areas of an inscribed and a circumscribed regular polygon of half the number of sides.

105. The area of a circumscribed regular polygon is an harmonic mean between the areas of an inscribed regular polygon of the same number of sides, and of a circumscribed regular polygon of half that number.

106. The perimeter of a circumscribed regular triangle is double that of the inscribed regular triangle.

107. The square described about a circle is four-thirds the inscribed dodecagon.

108. Two sides of a triangle are 3 and 12, and the included angle is 30°. Find the hypotenuse of an isosceles right triangle of equal area.

PLANE SAILING.

109. **Plane Sailing** is that branch of Navigation in which the surface of the earth is considered a plane. The problems which arise are therefore solved by the methods of Plane Trigonometry.

The following definitions will explain the technical terms which are employed :

The *difference of latitude* of two places is the arc of a meridian comprehended between the parallels of latitude passing through those places.

The *departure* between two meridians is the arc of a parallel of latitude comprehended between those meridians. It evidently diminishes as the distance from the equator at which it is measured increases.

When a ship sails in such a manner as to cross successive meridians at the same angle, it is said to sail on a *rhumb-line.* The constant angle which this line makes with the meridians is called the *course,* and the *distance* between two places is measured on a rhumb-line.

If we neglect the curvature of the earth, and consider the distance, departure, and difference of latitude of two places to be straight lines, lying in one plane, they will form a right triangle, called *the triangle of plane sailing.* If ABD be a plane triangle, right-angled at D, and AD represent the difference of latitude of A and B, DAB will be the course from

A to B, AB the distance, and DB the departure, measured from B, between the meridian of A and that of B.

110. Taking the earth's equatorial diameter to be 7925.6 miles, find the length in feet of the arc of one minute of a great circle.*

111. A ship sails from latitude 43° 45′ S., on a course N. by E., 2345 miles. Find the latitude reached, and the departure made.

112. A ship sails from latitude 1° 45′ N., on a course S.E. by E., and reaches latitude 2° 31′ S. Find the distance, and the departure.

113. A ship sails from latitude 13° 17′ S., on a course N.E. by E. $\frac{3}{4}$ E., until the departure is 207 miles. Find the distance, and the latitude reached.

114. A ship sails on a course between S. and E., 244 miles, leaving latitude 2° 52′ S., and reaching latitude 5° 8′ S. Find the course, and the departure.

115. A ship sails from latitude 32° 18′ N., on a course between N. and W., making a distance of 344 miles, and a departure of 103 miles. Find the course, and the latitude reached.

116. A ship sails on a course between S. and E., making a difference of latitude 136 miles, and a departure 203 miles. Find the distance, and the course.

117. A ship sails due north 15 *statute* miles an hour, for one day. What is the distance, in a straight line, from the point left to the point reached? (Take earth's radius, 3962.8 statute miles.)

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* The length of the arc of one minute of a great circle of the earth is called a *geographical mile*, or a *knot*. In the following problems, this is the distance meant by the term "mile," unless otherwise stated.

PARALLEL AND MIDDLE LATITUDE SAILING.

118. The difference of longitude of two places is the angle at the pole made by the meridians of these two places; or, it is the arc of the equator comprehended between these two meridians.

119. In **Parallel Sailing,** a vessel is supposed to sail on a parallel of latitude; that is, either due east or due west. The distance sailed is, in this case, evidently the departure made; and the difference of longitude made depends on the solution of the following problem:

120. Given the departure between any two meridians at any latitude, find the angle which those meridians make, or the difference of longitude of any point on one meridian from any point on the other. (The earth is considered to be a perfect sphere, and the solution depends on simple geometric and trigonometric principles. *Cf.* Problem 7.) The solution gives the following formula:

$$\text{Diff. long.} = \text{depart.} \times \text{sec. lat.}$$

121. A ship in latitude 42° 16′ N., longitude 72° 16′ W., sails due east a distance of 149 miles. What is the position of the point reached?

122. A ship in latitude 44° 49′ S., longitude 119° 42′ E., sails due west until it reaches longitude 117° 16′ E. Find the distance made.

123. In **Middle Latitude Sailing,** the departure between two places, not on the same parallel of latitude, is considered to be, approximately, the departure between the meridians of those places, measured on that parallel of latitude which lies midway between the parallels of the two places. Except in very high latitudes or excessive runs, such an assumption produces no great error. By the formula of Art. 120, then, we shall have —

$$\text{Diff. long.} = \text{depart.} \times \text{sec. mid. lat.}$$

124. A ship leaves latitude 31° 14′ N., longitude 42° 19′ W., and sails E.N.E. 325 miles. Find the position reached.

125. Find the bearing and distance of Cape Cod from Havana. (Cape Cod, 42° 2′ N., 70° 3′ W.; Havana, 23° 9′ N., 82° 22′ W.)

126. Leaving latitude 49° 57′ N., longitude 15° 16′ W., a ship sails between S. and W. till the departure is 194 miles, and the latitude is 47° 18′ N. Find the course, distance, and longitude reached.

127. Leaving latitude 42° 30′ N., longitude 58° 51′ W., a ship sails S.E. by S. 300 miles. Find the position reached.

128. Leaving latitude 49° 57′ N., longitude 30° W., a ship sails S. 39° W., and reaches latitude 47° 44′ N. Find the distance, and longitude reached.

129. Leaving latitude 37° N., longitude 32° 16′ W., a ship sails between N. and W. 300 miles, and reaches latitude 41° N. Find the course, and longitude reached.

130. Leaving latitude 50° 10′ S., longitude 30° E., a ship sails E.S.E., making 160 miles' departure. Find the distance, and position reached.

131. Leaving latitude 49° 30′ N., longitude 25° W., a ship sails between S. and E. 215 miles, making a departure of 167 miles. Find the course, and position reached.

132. Leaving latitude 43° S., longitude 21° W., a ship sails 273 miles, and reaches latitude 40° 17′ S. What are the *two* courses and longitudes, either one of which will satisfy the data?

133. Leaving latitude 17° N., longitude 119° E., a ship sails 219 miles, making a departure of 162 miles. What four sets of answers do we get?

134. A ship in latitude 30° sails due east 360 statute miles. What is the shortest distance from the point left to the point reached?

Solve the same problem for latitude 45°, 60°, etc.

TRAVERSE SAILING.

135. **Traverse Sailing** is the application of the principles of Plane and Middle Latitude Sailing to cases when the ship sails from one point to another on two or more different courses. Each course is worked up by itself, and these independent results are combined, as may be seen in the solution of the following example:

136. Leaving latitude 37° 16′ S., longitude 18° 42′ W., a ship sails N.E. 104 miles, then N.N.W. 60 miles, then W. by S. 216 miles. Find the position reached, and its bearing and distance from the point left.

We have, for the first course, difference of latitude 73.5 N., departure 73.5 E.

We have, for the second course, difference of latitude, 55.4 N., departure 23 W.

We have, for the third course, difference of latitude 42.1 S., departure 211.8 W.

On the whole, then, the ship has made 128.9 miles of north latitude, and 42.1 miles of south latitude. The place reached is therefore on a parallel of latitude 86.8 miles to the north of the parallel left; that is, in latitude 35° 49′.2 S.

The departure is, in the same way, found to be 161.3 miles W.; and the middle latitude is 36° 32′.6. With these data, and the formula of Art. 126, we find the difference of longitude to be 201 miles, or 3° 21′ W. Hence the longitude reached is 22° 3′ W.

With the difference of latitude 86.8 miles, and the departure 161.3 miles, we find the course to be N. 61° 43′ W., and the distance 183.2 miles. The ship has reached the same point that it would have reached, if it had sailed directly on a course N. 61° 43′ W., for a distance of 183.2 miles.

137. A ship leaves Cape Cod (Ex. 125), and sails S.E. by S. 114 miles, N. by E. 94 miles, W.N.W. 42 miles. Solve as in Art. 136.

138. A ship leaves Cape of Good Hope (latitude 34° 22′ S., longitude 18° 30′ E.), and sails N.W. 126 miles, N. by E. 84 miles, W.S.W. 217 miles. Solve as in Ex. 136.

EXAMINATION PAPERS.*

PLANE TRIGONOMETRY.

I.

(*Harvard College, Admission. June*, 1881. *Time*, 1½ *hours.*)

1. Define a logarithm. What is the logarithm of ½ in the system of which 16 is the base? Find the logarithm of 25 in the same system.

2. Compute the value of $\sqrt[5]{\frac{(0.012)^3 \times 0.27}{(64)^2 \times 0.00651}}$ by logarithms.

3. Find the functions of 127° 10′ from your trigonometric tables.

4. Prove the formula

$$(\cos A - \cos B)^2 + (\sin A - \sin B)^2 = 4 \sin^2 \frac{A-B}{2}.$$

5. Two sides of a triangle are 243 feet and 188 feet, and the angle opposite the second side is 42° 20′. Solve the triangle completely.

6. A pine tree growing on the side of a mountain, which is inclined to the horizontal at an angle of 20°, is broken by the wind but not severed at a distance of 40 feet from the ground. The top falls toward the foot of the mountain, and strikes the ground 50 feet from the base of the tree; find the height of the tree.

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* NOTE. In these papers, as in many text-books, the Greek letters α (*alpha*), β (*bayta*), γ (*gamma*), δ (*delta*), θ (*thayta*), ϕ (*phee*), are occasionally used to denote angles.

II.

(*Harvard College, Admission. June*, 1882. *Time*, 1½ *hours.*)

1. Explain the reason of the rule for finding the *characteristic* (or integral part) of the logarithm of a number.

Show that (according to this rule) the *mantissa* (or fractional part) is always *positive.*

In what cases is the logarithm, as a whole, positive, and in what cases negative?

Thus, state clearly the value of the logarithm of 36,270; of 0.003627. What decimal must be added to the latter logarithm to produce the logarithm of 0.01?

2. Find the time required to increase a sum of money a hundred fold, at *ten* per cent per annum, compound interest, payable yearly.

3. Find the formulas for the trigonometric functions of $90° + a$.

4. Find by the tables the logarithms of the trigonometric functions of $290° 38'$ (marking the signs).

5. An observer from a ship saw two headlands. The first bore E.N.E. (*i.e.* $67° 30'$ from N. towards E.), and the second N.W. by N. (*i.e.* $33° 45'$ from N. towards W.). After he had sailed 16.25 miles N. by W. (*i.e.* $11° 15'$ from N. towards W.), the first headland bore due E., and the second N.W. by W. (*i.e.* $56° 15'$ from N. towards W.). Find the direction and distance of the second headland from the first.

6. Prove the formulas:

$$\frac{\cos a - \cos \beta}{\cos a + \cos \beta} = -\tan \tfrac{1}{2}(a + \beta) \tan \tfrac{1}{2}(a - \beta),$$

$$\sin \theta = \frac{2 \tan \frac{1}{2}\theta}{1 + \tan^2 \frac{1}{2}\theta}.$$

III.

(*Harvard College, Freshman Examination. April,* 1879. *Time,* 3 *hours.*)

1. Prove the relations between the sine, cosine, and tangent of $90° + \phi$ and the functions of ϕ. Draw a figure for the case where ϕ is obtuse, and show that the proof still holds good.

Confirm your results by means of the formulas for the sine and cosine of the sum of the two angles.

2. Deduce formulas for the sine, cosine, and tangent of $2a$ and $\frac{1}{2}a$, in terms of functions of a.

3. Prove the formula:

$$\cos(\alpha + \beta) \sin\beta - \cos(\alpha + \gamma) \sin\gamma =$$
$$\sin(\alpha + \beta) \cos\beta - \sin(\alpha + \gamma) \cos\gamma.$$

4. Prove that in any triangle

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

5. Deduce the formulas for the tangents of the half angles of a triangle, in terms of the sides.

6. Solve the triangles:

$$C = 35°, \qquad a = 500, \qquad c = 250,$$
$$B = 22° \, 22', \qquad a = 67.06, \qquad b = 60.03.$$

7. The Delta measures 241 yds. on Cambridge St. and 115 yds. on Quincy St., and the angle between these streets is 88° 52′. Find the other angles of the Delta.

8. Find the area of the Delta.

9. A person travelling east in a railroad train observes a tower situated south of a station A, and on the same horizontal plane with the railroad. At the station B he finds that the distance of the tower is 2 miles; and at C, 3 miles from B, its distance is 4 miles. Find the distance of the tower from A.

IV.

(*Harvard College, Freshman Examination. April,* 1880. *Time,* 3 *hours.*)

1. Deduce the formula $\sin(\alpha+\beta)=\ldots\ldots$, drawing the figure for the case in which α is in the second quadrant and $\alpha+\beta$ in the third.

2. Deduce the formulas for $\cos 2\alpha$, $\sin\frac{1}{2}\alpha$, and $\cos\frac{1}{2}\alpha$, in terms of functions of α.

3. Prove the theorem of the sines.

4. From the fundamental formulas deduce the formula

$$\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}=\frac{a+b}{a-b}.$$

5. Prove that

$$\frac{\cos(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma}=1-\tan\alpha\tan\beta-\tan\beta\tan\gamma-\tan\gamma\tan\alpha.$$

6. In a triangle $B=4^\circ 13.4'$ and $a=2001$, give all the solutions in the following cases:

(1) $b=150$,
(2) $b=200$,
(3) $b=2001$.

7. A, B, and C are the corners of a triangular field. A is 40 ft. W. of B and 400 ft. S.W. of C. What is the area of the field? What is the length of the fence which encloses it?

8. From two corners of the Delta, A and B, lines which make angles of $19^\circ 52'$ and $57^\circ 32'$ respectively with the side AB meet directly under Memorial Hall tower. The length of AB is 345.1 ft., and the angle of elevation of the tower at A is $32^\circ 26'$. Find the height of the tower, and its angle of elevation at B.

V.

(*Harvard College, Freshman Examination. April,* 1881. *Time,* 3 *hours.*)

1. Deduce the formulas which connect the functions of $(90^\circ + \phi)$ and ϕ.

2. Prove the fundamental formula for $\cos(\alpha + \beta)$.

3. From the formula just found, obtain three values for $\cos 2\alpha$.

4. Find the values of $\sin\frac{\alpha}{2}$ and $\cos\frac{\alpha}{2}$, in terms of $\cos\alpha$.

5. Prove the formula

$$(\cos A - \cos B)^2 + (\sin A - \sin B)^2 = 4\sin^2\frac{A-B}{2}.$$

6. Solve the following triangles:

$b = 2434$, $c = 1881$, $\mathrm{C} = 42^\circ\, 22'$.
$a = 0.00543$, $c = 0.07003$, $\mathrm{A} = 4^\circ\, 27'$.

7. The sides of a triangle are 715, 541, and 368; find one angle and the area.

8. The height of Memorial Hall tower is 190 feet. From its top the angles of depression of the corners of the Delta which lie on Cambridge St. are 57° 44′ and 16° 59′, and the angle subtended by the line joining these corners is 99° 30′. Find the length of the Delta on Cambridge St.

VI.

(*Harvard College, Freshman Examination. April,* 1882. *Time,* 3 *hours.*)

1. Obtain the formulas which connect the sine, cosine, and tangent of $(180^\circ + \phi)$ with the functions of ϕ.

2. Assuming the formulas for the sine and cosine of the sum of two angles, prove that

(1) $\tan(\alpha + \beta) = \underline{\hspace{6em}}$
(2) $\sin\frac{1}{2}\alpha = \sqrt{\frac{1}{2}(1 - \cos\alpha)}$.

3. Find all the values of x, between 0° and 360°, which will satisfy the equations

$$(1)\quad \tan x = 2 \sin 2x,$$
$$(2)\quad (\sin x + \cos x)^2 = 2 \sin 2x.$$

4. The length of each side of a regular dodecagon is 24 feet; find the radius of the inscribed circle and the area of the polygon.

5. In a certain triangle, $a = 20$, $B = 3° 24'$, $C = 85° 31'$. Find c by aid of the table containing the values of S and T.

6. Prove the Theorem of Sines, and solve the triangles

(1) $b = 468$, $c = 327$, $C = 34° 15'$;
(2) $a = 0.003641$, $c = 0.08091$, $A = 5° 20'$.

7. Given $a^2 = b^2 + c^2 - 2bc \cos A$; obtain the formula

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

The sides of a triangle are $a = 2408$, $b = 2028$, $c = 1884$; find the angle A.

8. Hingham is 12 miles south-east of Boston; Quincy is $5\frac{3}{4}$ miles west of Hingham. How far is Quincy from Boston?

9. A person ascending Memorial Hall tower stops to rest at a window, and notices that the angle of elevation of the vane on Appleton Chapel is 3° 34′. When he reaches the top of the tower, 190 feet above the ground, he finds that the height of the Chapel subtends an angle of 13° 9′. The horizontal distance between the two towers being 492 feet, find the height of Appleton Chapel and the distance of the window above the ground.

VII.

(*Cambridge, Eng. 2nd Previous Exam., Dec.* 7, 1876. *Time,* 2½ *hours.*)

1.* Assuming that the angle subtended at the centre of any circle by an arc equal to its radius is a constant angle, show that any angle may be expressed by the fraction $\frac{\text{arc}}{\text{radius}}$, the constant angle being taken as the unit.

Find the length of the arc subtended by an angle of 60° in a circle whose radius is 3 feet.

2. The sine of a certain angle is $\frac{1}{4}$; find the other trigonometrical ratios of the angle.

3. Trace the change in sign and magnitude in the tangent of an angle, as the angle increases from 0° to 360°.

4. Find, by a geometrical construction, the cosine of 60° and of 45°, and deduce the value of $\cos 3360°$ and $\cos 2565°$.

5. Prove the formulas:

(1) $\sin(A-B)=\sin A\cos B-\cos A\sin B$,

(2) $\cos 2A=\dfrac{1-\tan^2 A}{1+\tan^2 A}$,

(3) $\dfrac{\sin 2A+\sin 4A}{\cos 2A+\cos 4A}=\tan 3A$.

6. Express the cosine of half an angle in terms of the sine of the angle, and explain the double sign.

Employ the formula to find the value of $\cos 75°$, having given $\sin 150°=\frac{1}{2}$.

7. If A, B, C be the angles of a triangle, and a, b, c the sides respectively opposite to them, show that

$$\cos\tfrac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}},$$

where $s=$ one-half the sum of the sides.

8. Find the greatest angle in a triangle whose sides are 7 feet, 8 feet, and 9 feet.

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* For aid in solving this and similar questions, see Wentworth & Hill's Tables, pages xi and xii.

VIII.

(*Cambridge, Eng. 2nd Previous Exam., Dec.* 7, 1877. *Time,* $2\frac{1}{2}$ *hours.*)

1. Define the cosine, cotangent, and cosecant of an angle, and prove that these ratios remain unchanged so long as the angle is the same.

Find the value of these three ratios for an angle of 45°.

2. Prove the formulas:

$$(1) \quad \sin A = \sqrt{1 - \cos^2 A},$$

$$(2) \quad \cos A = \frac{1}{\sqrt{1 + \tan^2 A}}.$$

If $\sec A = \sqrt{2}$, find $\tan A$. *Ans.* 1.

3. Prove that

$$\sin(90^\circ + A) = \cos A, \text{ and } \cos(90^\circ + A) = -\sin A.$$

Hence show that $\cos(180^\circ + A) = -\cos A$.

4. Show that $\cos^2 A \tan^2 A + \sin^2 A \cot^2 A = 1$.

5. Prove that $\cos(A + B) = \cos A \cos B - \sin A \sin B$. Hence show that

$$\frac{\cos(A + B + C)}{\sin A \sin B \sin C} = \cot A \cot B \cot C - \cot A - \cot B - \cot C.$$

6. Given that $\sin\frac{1}{2}A = \frac{3}{5}$, find the value of $\tan A$.
Ans. $\frac{24}{7}$.

7. Prove that the sides of any plane triangle are proportional to the sines of the angles opposite to these sides.

If $2s =$ the sum of the three sides (a, b, c) of a triangle, and if A be the angle opposite to the side a, prove that

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

8. Prove that in any plane triangle

$$\tan\tfrac{1}{2}(A - B) = \frac{a - b}{a + b}\cot\tfrac{1}{2}C.$$

9. If the side a and the angles A and B of a triangle be known, prove that the side b may be found by means of the formula,

$$\log b = \log a + \log \sin B - \log \sin A.$$

Find b, having given that $a = 1000$ yards, $A = 50°$, $B = 64°$.

Ans. 1173.29 yards.

10. The minute-hand of a clock is 3 feet 6 inches in length; find how far its point will move in a quarter of an hour, it being assumed that $\pi = \frac{22}{7}$. *Ans.* 5 feet 6 inches.

IX.

(*Cambridge, Eng. 2nd Previous Exam., Dec.* 10, 1878. *Time,* $2\frac{1}{2}$ *hours.*)

1. Define sine, cotangent; and prove that $\sin^2 A + \cos^2 A = 1$. Express the other trigonometrical ratios in terms of the cosine.

2. What is meant by the circular measure of an angle? How is the number of degrees in an angle found from its circular measure? How many degrees are in the unit of circular measure?

3. Prove that

$$(1) \quad \sin(180° + A) = -\sin A,$$

$$(2) \quad \tan(90° + A) = -\cot A.$$

What is the use of these equations?

4. Find the general form of all the angles whose sine is the same as $\sin \theta$.

Write down the sines of all the angles which are multiples of 30° and less than 360°.

5. Prove the following relations:

(1) $\cos(A-B)=\cos A\cos B+\sin A\sin B$,

(2) $\sin A+\sin B=2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$,

(3) $\tan^2 A=\dfrac{1-\cos 2A}{1+\cos 2A}$.

6. Find $\cos 30°$, $\tan 45°$, $\sin 15°$.

7. If $\tan A+\sec A=2$, prove that $\sin A=\frac{3}{5}$ when A is less than 90°.

If $\sin A=\frac{4}{5}$, prove that $\tan A+\sec A=3$ when A is less than 90°.

8. Prove that $\cos 3A=4\cos^3 A-3\cos A$, and find $\tan 3A$ in terms of $\tan A$.

9. In a triangle A, B, C, whose sides are respectively a, b, c, prove that

(1) $\sin\frac{1}{2}A=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$,

(2) $\dfrac{\sin A}{a}=\dfrac{\sin B}{b}=\dfrac{\sin C}{c}$.

10. Solve a triangle, having given two sides and the angle opposite one of them.

Find A, B, b, having given $a=25$, $c=24$, $C=65°\,59'$.

X.

(*Cambridge, Eng. 2nd Previous Exam., Dec.* 10, 1879. *Time,* 2½ *hours.*)

1. Define 1°. Assuming that $\frac{22}{7}$ is the circular measure of two right angles, express the angle $A°$ in circular measure.

2. Define the sine, secant, and cotangent of an angle, and express any two of these ratios in terms of the third.

Find the trigonometrical ratios of the angle whose cosine is $\frac{3}{5}$.

3. Prove that

(1) $\cos(180^\circ + A) = \cos(180^\circ - A)$,

(2) $\tan(90^\circ + A) = \cot(180^\circ - A)$.

4. Express the cosine of the difference of two angles in terms of the sines and cosines of these angles.

Prove that

$$*\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}.$$

5. Prove the formulas:

(1) $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$,

(2) $\sin\frac{1}{2}x + \cos\frac{1}{2}x = \pm\sqrt{1+\sin x}$,

(3) $\sin x(2\cos x - 1) = 2\sin\frac{1}{2}x\cos\frac{3}{2}x$.

6. Trace the changes in sign and magnitude of

$$\frac{2\sin\theta - \sin 2\theta}{2\sin\theta + \sin 2\theta},$$

as θ changes from 0 to 2π.

7. Express the cosine of any angle of a triangle in terms of the sides of the triangle.

If the angle opposite the side a be 60°, and if b, c be the remaining sides of the triangle, prove that

$$(a+b+c)(b+c-a) = 3bc.$$

8. Solve a triangle, having given the three sides.

Given $A = 36^\circ$, $B = 72^\circ$, and $a = 1$; solve the triangle.

9. The sides of a triangle are 2, 3, 4; find the least angle.

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* $\tan^{-1}x$ = arc whose tangent is x.

XI.

(*Cambridge, Eng. 2nd Previous Exam., Dec.* 10, 1880. *Time,* $2\frac{1}{2}$ *hours.*)

1. Express in degrees, minutes, etc., (i.) the angle whose circular measure is $\frac{1}{20}\pi$; (ii.) the angle whose circular measure is 5.

If the angle subtended at the centre of a circle by the side of a regular pentagon be the unit of angular measurement, by what number is a right angle represented?

2. Find, by geometrical constructions, the cosine of 45° and the sine of 120°.

Prove that

$$(\sin 30° + \cos 30°)(\sin 120° + \cos 120°) = \sin 30°.$$

3. If $\csc A = 9$, find $\cot A$ and $\sec A$.

4. Prove that

$$\cos(180° + A) = -\cos A.$$

Find the value of (i.) $\cot 840°$; (ii.) $\sec 3\pi$.

5. Assuming the formula for the sine of the sum of two angles in terms of the sines and cosines of the separate angles, find (i.) $\sin 75°$; (ii.) $\sin 3A$ in terms of $\sin A$.

6. Prove the formulas:

(1) $\cos^2(A - B) - \sin^2(A + B) = \cos 2A \cos 2B$,

(2) $1 + \tan x \tan \frac{1}{2}x = \sec x$.

7. Prove that

$$\cos A + \cos B = 2\cos\tfrac{1}{2}(A + B)\cos\tfrac{1}{2}(A - B),$$

and express $\dfrac{\cos 5\theta + \cos\theta}{\cos 5\theta - \cos\theta}$ as a single term.

8. Solve the equations:

(1) $5\tan^2 x + \sec^2 x = 7$,

(2) $\cos 5\theta + \cos 3\theta = \sqrt{2}\cos 4\theta$.

9. Prove that in any triangle $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$.
Obtain the formula for $\tan \frac{1}{2} A$ in terms of the sides.

10. Find an expression for the area of a triangle in terms of its sides. The lengths of the sides of a triangle are 3 feet, 5 feet, and 6 feet; what is its area?

11. Given that

$$\sin 38^\circ 25' = 0.6213757, \quad \sin 38^\circ 26' = 0.6216036;$$

find the angle whose sine is 0.6215000.

XII.

(*Cambridge, Eng. 2nd Previous Exam.*, *Dec.* 10, 1881. *Time*, $2\frac{1}{2}$ *hours.*)

1. Define the unit of circular measure. The ratio of the circumference of a circle to its diameter being 3.14159, find the circular measure of an angle of 126°.

2. Define the tangent, cotangent, and cosecant of an angle. Find the tangent and cotangent of an angle whose cosecant is 1.25.

3. Trace the changes in sign and magnitude of $\sin A$ as A changes from 90° to 270°.

4. Prove the following:

$$(1) \quad \tan(\pi + A) = \tan A,$$

$$(2) \quad \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B},$$

$$(3) \quad \frac{\sin A + \sin B}{\cos A - \cos B} = \cot \frac{B - A}{2}.$$

5. Determine the value of $\cos 18°$, and prove that

$$\cos 36° = \cos 60° + \cos 72°.$$

6. Show that for certain values of the angles

$$2 \cos \tfrac{1}{2} A = \sqrt{1 + \sin A} - \sqrt{1 - \sin A}.$$

Is this formula true for values of A lying between 200° and 220°? and if not, how must it be modified?

7. Prove that in any triangle, with the usual notation,

$$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}},$$

and that the area is equal to

$$\sqrt{s(s-a)(s-b)(s-c)}.$$

Show, also, that

$$\sin^2 A = \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C.$$

8. When one side of a triangle and the two adjacent angles are given, show how to solve the triangle.

Find the greatest side of the triangle, of which one side is 2183 feet, and the adjacent angles are 78° 14′ and 71° 24′.

SPHERICAL TRIGONOMETRY.

CHAPTER V.

THE RIGHT SPHERICAL TRIANGLE.

§ 45. INTRODUCTION.

The object of *Spherical Trigonometry* is to show how spherical triangles are solved. To *solve* a spherical triangle is to compute any three of its parts when the other three parts are given.

The sides of a spherical triangle are arcs of great circles. They are measured in degrees, minutes, and seconds, and therefore by the plane angles formed by radii of the sphere drawn to the vertices of the triangle. Hence, their measures are independent of the length of the radius, which may be assumed to have any convenient numerical value; as, for example, unity.

The angles of the triangle are measured by the angles made by the planes of the sides. Each angle is also measured by the number of degrees in the arc of a great circle, described from the vertex of the angle as a pole, and included between its sides.

The sides may have any values from 0° to 360°; but in this work only sides that are less than 180° will be considered. The angles may have any values from 0° to 180°.

If any two parts of a spherical triangle are either both less than 90° or both greater than 90°, they are said to be **alike in kind**; but if one part is less than 90°, and the other part greater than 90°, they are said to be **unlike in kind.**

Spherical triangles are said to be isosceles, equilateral, equiangular, right, and oblique, under the same conditions as plane triangles. A *right* spherical triangle, however, may have one, two, or three right angles.

When a spherical triangle has one or more of its sides equal to a quadrant, it is called a **quadrantal triangle.**

It is shown in Solid Geometry, that in every spherical triangle

I. *The sum of the sides is less than* 360°.

II. *The sum of the angles is greater than* 180°.

III. *If, from the vertices as poles, arcs of great circles are described, another spherical triangle is formed so related to the first triangle that the sides of each triangle are supplements of the angles opposite to them in the other triangle.*

Two such triangles are said to be *polar* with respect to each other.

Let A, B, C (Fig. 35) denote the angles of one triangle; a, b, c the sides opposite these angles respectively; and let A', B', C' and a', b', c' denote the corresponding sides and angles of the polar triangle. Then the above theorem gives the six following equations:

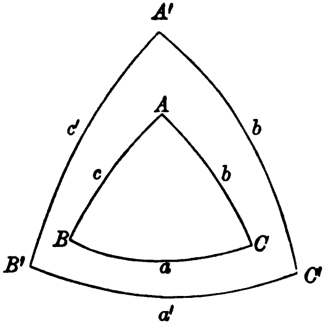

Fig. 35.

$$\begin{aligned}A + a' &= 180°,\\ B + b' &= 180°,\\ C + c' &= 180°,\\ A' + a &= 180°,\\ B' + b &= 180°,\\ C' + c &= 180°\end{aligned}$$

Exercise XIX.

1. The angles of a triangle are 70°, 80°, and 100°; find the sides of the polar triangle.

2. The sides of a triangle are 40°, 90°, and 125°; find the angles of the polar triangle.

3. Prove that the polar of a quadrantal triangle is a right triangle.

4. Prove that, if a triangle have three right angles, the sides of the triangle are quadrants.

5. Prove that, if a triangle have two right angles, the sides opposite these angles are quadrants, and the third angle is measured by the number of degrees in the opposite side.

6. How can the sides of a spherical triangle be found in units of length, when the length of the radius of the sphere is known?

7. Find the lengths of the sides of the triangle in Example 2, if the radius of the sphere is 4 feet.

§ 46. FORMULAS RELATING TO RIGHT TRIANGLES.

As is evident from § 45, Examples 4 and 5, the only kind of right triangle requiring further investigation is that which contains *only one* right angle.

Let ABC (Fig. 36) be such a right triangle, and let A, B, C denote the angles of the triangle; a, b, c, respectively, the opposite sides.

Let C be the right angle, and for the present suppose that each of the other parts is less than 90°.

Let planes be passed through the sides, intersecting in the radii OA, OB, and OC; and for the sake of simplicity let the radius of the sphere be taken equal to 1.

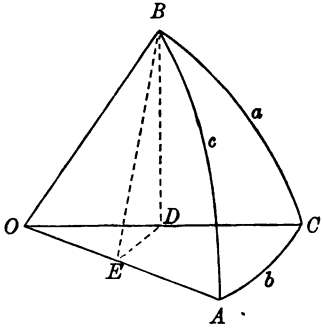

Fig. 36.

Also, let a plane perpendicular to OA be passed through B, cutting OA at E and OC at D. Join BE, BD, and DE.

BE and DE are each $\perp OA$ (Geom. § 454); therefore $\angle BED = A$. The plane BDE is $\perp$ the plane AOC (Geom. § 472); hence BD, which is the intersection of the planes BDE and BOC, is $\perp$ the plane AOC (Geom. § 475), therefore $\perp$ OC and DE. Now

$$\cos c = OE = OD \times \cos b,$$

and $OD = \cos a$.

$$\therefore \ \mathbf{\cos c = \cos a \cos b}. \qquad [37]$$

$$\sin a = BD = BE \times \sin A,$$

and $BE = \sin c$.

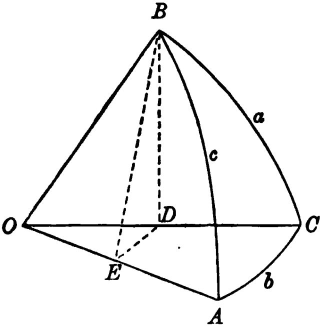

Fig. 36 (bis).

$$\begin{aligned} &\text{Therefore,} && \mathbf{\sin a = \sin c \sin A} \\ &\text{changing letters,} && \mathbf{\sin b = \sin c \sin B} \end{aligned} \Bigg\}. \qquad [38]$$

$$\begin{aligned} &\text{Again,} && DE = BE \times \cos A, \\ &\text{and also} && DE = OD \times \sin b; \\ &\text{hence,} && BE \times \cos A = OD \times \sin b; \\ &\text{that is,} && \sin c \cos A = \cos a \sin b; \\ &\text{whence} && \cos A = \cos a \sin b \csc c.^{*} \end{aligned}$$

By substituting in this formula the value of $\cos a$, obtained from [37], we obtain

$$\begin{aligned} & && \mathbf{\cos A = \tan b \cot c} \\ &\text{In like manner,} && \mathbf{\cos B = \tan a \cot c} \end{aligned} \Bigg\}. \qquad [39]$$

And by substituting in the same formula the value of $\sin b$, as given in [38], we obtain

$$\begin{aligned} & && \mathbf{\cos A = \cos a \sin B} \\ &\text{In like manner,} && \mathbf{\cos B = \cos b \sin A} \end{aligned} \Bigg\}. \qquad [40]$$

$$\begin{aligned} &\text{Also,} && BD = DE \times \tan A, \\ & && BD = \sin a, \\ & && DE = \cos a \sin b; \\ &\text{therefore,} && \sin a = \cos a \sin b \tan A; \end{aligned}$$

---

* To avoid fractions, $\csc c$ is written in place of its equal, $\frac{1}{\sin c}$.

whence, $\qquad \mathbf{\sin b = \tan a \cot A}$
and, similarly, $\mathbf{\sin a = \tan b \cot B}$

$$\left.\begin{aligned}\sin \mathrm{b} &= \tan \mathrm{a} \cot \mathrm{A}\\ \sin \mathrm{a} &= \tan \mathrm{b} \cot \mathrm{B}\end{aligned}\right\}. \qquad [41]$$

If in [37] we substitute for $\cos a$ and $\cos b$ their values from [40], we obtain

$$\cos \mathrm{c} = \cot \mathrm{A} \cot \mathrm{B}. \qquad [42]$$

In deducing these formulas, it has been assumed that all the parts of the triangle, except the right angle, are less than 90°. But the formulas also hold true when this hypothesis is not fulfilled.

Let one of the legs a be greater than 90°, and construct a figure for this case (Fig. 37) in the same manner as Fig. 36.

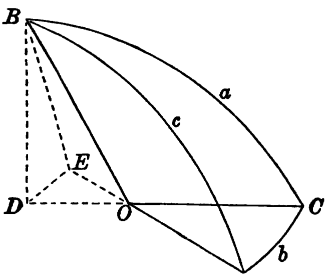

Fig. 37.

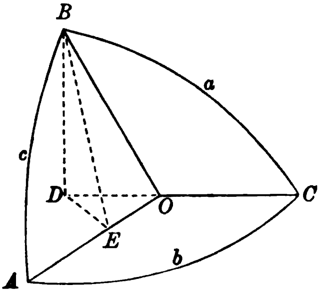

Fig. 38.

The auxiliary plane BDE will now cut both CO and AO produced beyond the centre O; and we have

$$\cos(180 - c) = OE = OD \times \cos b = \cos(180 - a) \cos b,$$

a result which reduces to [37] if we substitute $-\cos c$ in place of $\cos(180 - c)$ and $-\cos a$ in place of $\cos(180 - a)$. Likewise, the other formulas, [38]–[42], hold true in this case.

Again, suppose that both the legs a and b are greater than 90°. In this case the plane BDE (Fig. 38) will cut CO produced beyond O, and AO between A and O; and we have

$$\begin{aligned}\cos c &= OE = OD \times \cos(180 - b)\\ &= \cos(180 - a) \cos(180 - b)\\ &= \cos a \cos b,\end{aligned}$$

a result agreeing with [37]. And the remaining formulas may be easily shown to hold true.

Like results follow in all cases; in other words, Formulas [37]–[42] are universally true.

EXERCISE XX.

1. Prove, by aid of Formula [37], that the hypotenuse of a right triangle is *less than* or *greater than* 90°, according as the two legs are *alike* or *unlike* in kind.

2. Prove, by aid of Formula [40], that in a right spherical triangle each leg and the opposite angle are always alike in kind.

3. What inferences may be drawn respecting the values of the other parts: (i.) if $c = 90°$; (ii.) if $a = 90°$; (iii.) if $c = 90°$ and $a = 90°$; (iv.) if $a = 90°$ and $b = 90°$?

Deduce from [37]–[42] the following formulas:

4. $\tan^2 \tfrac{1}{2} b = \tan \tfrac{1}{2}(c - a) \tan \tfrac{1}{2}(c + a)$.
5. $\tan^2 (45° - \tfrac{1}{2} A) = \tan \tfrac{1}{2}(c - a) \cot \tfrac{1}{2}(c + a)$.
6. $\tan^2 \tfrac{1}{2} B = \sin (c - a) \csc (c + a)$.
7. $\tan^2 \tfrac{1}{2} c = - \cos (A + B) \sec (A - B)$.
8. $\tan^2 \tfrac{1}{2} a = \tan [\tfrac{1}{2}(A + B) - 45°] \tan [\tfrac{1}{2}(A - B) + 45°]$.

§ 47. NAPIER'S RULES.

The formulas deduced in § 46 express the relations between five parts of a right triangle, — the three sides and the two oblique angles. All these relations may be shown to follow from two very useful Rules, devised by Baron Napier, the inventor of Logarithms.

For this purpose the right angle (not entering the formulas) is left out of account, and instead of the hypotenuse and the two oblique angles, their respective *complements* are employed; so that the five parts considered by the Rules are: *a*, *b*, *co. c*, *co. A*, *co. B*. Any one of these parts may be called a *middle* part; and then the two parts immediately adjacent are called *adjacent* parts, and the other two are called *opposite* parts.

Rule I. *The sine of the middle part is equal to the product of the tangents of the adjacent parts.*

Rule II. *The sine of the middle part is equal to the product of the cosines of the opposite parts.*

These Rules are easily remembered by the expressions, **tan. ad.** and **cos. op.**

The correctness of these Rules may be shown by taking each of the five parts as middle part, and comparing the resulting equations with the equations contained in Formulas [37]–[42].

For example, let *co. c* be taken as middle part, then *co. A* and *co. B* are the adjacent parts, and a and b the opposite parts,—as is very plainly seen in Fig. 39, in which the order of the parts is shown by arranging them around the circumference of a circle. Then, by Napier's Rules:

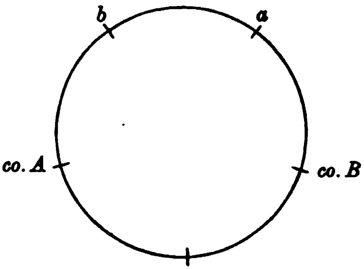

Fig. 39.

$$\sin(co.\ c) = \tan(co.\ A)\tan(co.\ B),$$

or

$$\cos c = \cot A\ \cot B;$$

$$\sin(co.\ c) = \cos a\ \cos b,$$

or

$$\cos c = \cos a\ \cos b;$$

results which agree with Formulas [37] and [42] respectively.

EXERCISE XXI.

1. Show that Napier's Rules lead to the equations contained in Formulas [38], [39], [40], and [41].

2. What will Napier's Rules become, if we take as the five parts of the triangle, the hypotenuse, the two oblique angles, and the *complements* of the two legs?

§ 48. SOLUTION OF RIGHT TRIANGLES.

By means of Formulas [37]–[42] we can solve a right triangle in all possible cases. In every case two parts besides the right angle must be given.

CASE I. *Given the two legs a and b.*

The solution is contained in Formulas [37] and [41]; viz:

$$\begin{aligned} \cos c &= \cos a \cos b, \\ \tan A &= \tan a \csc b, \\ \tan B &= \tan b \csc a. \end{aligned}$$

For example, let $a = 27° 28' 36''$, $b = 51° 12' 8''$; then the solution by logarithms is as follows:

$$\begin{array}{l} \log\cos a = 9.94802 \\ \underline{\log\cos b = 9.79697} \\ \log\cos c = 9.74499 \\ \qquad\qquad c = 56° 13' 40'' \end{array}$$

$$\begin{array}{l|l} \log\tan a \ = 9.71604 & \log\tan b \ = 10.09477 \\ \underline{\log\csc b \ = 0.10826} & \underline{\log\csc a \ = \ \ 0.33593} \\ \log\tan A = 9.82430 & \log\tan B = 10.43070 \\ \qquad\quad A = 33° 42' 50'' & \qquad\quad B = 69° 38' 54'' \end{array}$$

CASE II. *Given the hypotenuse c and the leg a.*

From Formulas [37], [38], and [39] we obtain

$$\begin{aligned} \cos b &= \cos c \sec a, \\ \sin A &= \sin a \csc c, \\ \cos B &= \tan a \cot c. \end{aligned}$$

Although two angles in general correspond to $\sin A$, one acute the other obtuse, yet in this case the indetermination is removed by the fact that A and a must be alike in kind (see Exercise XX., Example 2).

CASE III. *Given the leg a and the opposite angle A.*

By means of Formulas [38], [40], and [41], we find, that

$$\begin{aligned}\sin c &= \sin a \csc A,\\ \sin b &= \tan a \cot A,\\ \sin B &= \sec a \cos A;\end{aligned}$$

or, from [37] and [39],

$$\begin{aligned}\cos b &= \cos c \sec a,\\ \cos B &= \tan a \cot c.\end{aligned}$$

When c has been computed, b and B are determined by these values of their cosines; but, since c must be found from its sine, c may have in general two values which are supplements of one another. This case, therefore, really admits of two solutions.

CASE IV. *Given the leg a and the adjacent angle B.*

Formulas [39], [40], and [41] give

$$\begin{aligned}\tan c &= \tan a \sec B,\\ \tan b &= \sin a \tan B,\\ \cos A &= \cos a \sin B.\end{aligned}$$

CASE V. *Given the hypotenuse c and the oblique angle A.*

From Formulas [38], [39], and [42] it follows that

$$\begin{aligned}\sin a &= \sin c \sin A,\\ \tan b &= \tan c \cos A,\\ \cot B &= \cos c \tan A.\end{aligned}$$

Here a is determined by $\sin a$, since a and A must be alike in kind (see Exercise XX., Example 2).

CASE VI. *Given the two oblique angles A and B.*

By means of Formulas [40] and [42] we obtain

$$\begin{aligned}\cos c &= \cot A \cot B,\\ \cos a &= \cos A \csc B,\\ \cos b &= \cos B \csc A.\end{aligned}$$

NOTE 1. In Case I. (a and b given) the formula for computing c fails to give accurate results when c is very near 0° or 180°; in this case it may be found with greater accuracy by first computing B, and then computing c, as in Case IV.

NOTE 2. In Case II. (c and a given), if b is very near 0° or 180°, it may be computed more accurately by means of the derived formula

$$\tan^2 \tfrac{1}{2} b = \tan \tfrac{1}{2} (c + a) \tan \tfrac{1}{2} (c - a).$$

And if A is so near 90° that it cannot be found accurately in the Tables it may be computed from the derived formula

$$\tan^2 (45^\circ - \tfrac{1}{2} A) = \tan \tfrac{1}{2} (c - a) \cot \tfrac{1}{2} (c + a).$$

In like manner, when B cannot be accurately found from its cosine we may make use of the formula

$$\tan^2 \tfrac{1}{2} B = \sin (c - a) \csc (c + a).$$

NOTE 3. In Case III. (a and A given), when the formulas for the required parts do not give accurate results, we may employ the derived formulas

$$\begin{aligned} \tan^2(45^\circ - \tfrac{1}{2} c) &= \tan \tfrac{1}{2}(A - a) \cot \tfrac{1}{2}(A + a), \\ \tan^2(45^\circ - \tfrac{1}{2} b) &= \sin (A - a) \csc (A + a), \\ \tan^2(45^\circ - \tfrac{1}{2} B) &= \tan \tfrac{1}{2}(A - a) \tan \tfrac{1}{2}(A + a). \end{aligned}$$

NOTE 4. In Case IV. (a and B given), if A is near 0° or 180°, it may be more accurately found by first computing b and then finding A.

NOTE 5. In Case V. (c and A given), if a is near 90°, it may be found by first computing b, and then computing a by means of Formula [41].

NOTE 6. In Case VI. (A and B given), for unfavorable values of the sides greater accuracy may be obtained by means of the derived formulas

$$\begin{aligned} \tan^2 \tfrac{1}{2} c &= -\cos (A + B) \sec (A - B), \\ \tan^2 \tfrac{1}{2} a &= \tan [\tfrac{1}{2} (A + B) - 45^\circ] \tan [45^\circ + \tfrac{1}{2} (A - B)], \\ \tan^2 \tfrac{1}{2} b &= \tan [\tfrac{1}{2} (A + B) - 45^\circ] \tan [45^\circ - \tfrac{1}{2} (A - B)]. \end{aligned}$$

NOTE 7. In Cases I., IV., and V., the solution is always possible; in Case II., in order that the solution should be possible, it is necessary and sufficient that $\sin a < \sin c$; in Cases III. and VI., the conditions of possibility follow obviously from the equations employed; in Case III., it is also necessary that a and A should be alike in kind.

NOTE 8. It is easy to trace analogies between the formulas for solving right spherical triangles and those for solving right plane triangles. The former, in fact, become identical with the latter if we suppose the radius of the sphere to be infinite in length; in which case the cosines of the sides become each equal to 1, and the ratios of the sines of the sides and of the tangents of the sides must be taken as equal to the ratios of the sides themselves.

If the formula required for any case is not remembered, it is always easy to find it by means of Napier's Rules. In applying these Rules we must choose for the middle part that one of the three parts considered—the two given and the one required—which will make the other two either adjacent parts or opposite parts.

For example: *given a and B; solve the triangle.*

First, represent the parts as in Fig. 40, and to prevent mistakes mark each of the given parts with a cross. To find b, take a as the middle part; then b and $co. B$ are adjacent parts; and by Rule I.,

$$\sin a = \tan b \cot B;$$

whence, $\tan b = \sin a \tan B$.

To find c, take $co. B$ as middle part; then a and $co. c$ are adjacent parts; and by Rule I.,

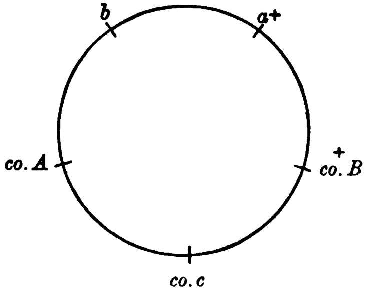

Fig. 40.

$$\cos B = \tan a \cot c;$$

whence, $\tan c = \tan a \sec B$.

To find A, take $co. A$ as middle part; then a and $co. B$ are the opposite parts; and by Rule II.,

$$\cos A = \cos a \sin B.$$

In like manner, every case of a right spherical triangle may be solved.

EXERCISE XXII.

Solve the following right triangles, taking for the given parts in each case those printed in columns I. and II.:

	I.	II.	III.	IV.	V.
	a	*b*	*c*	*A*	*B*
1	36° 27′	43° 32′ 31″	54° 20′	46° 59′ 43.2″	57°59′19.3″
2	86° 40′	32° 40′	87° 11′ 39.8″	88° 11′ 57.8″	32°42′38.7″
3	50°	36° 54′ 49″	59° 4′ 25.7″	63° 15′ 13.1″	44°26′21.6″
4	120° 10′	150° 59′ 44″	63° 55′ 43.2″	105° 44′ 21.25″	147°19′47.14″
	c	*a*	*b*	*A*	*B*
5	55° 9′ 32″	22° 15′ 7″	51° 53′	27° 28′ 25.7″	73° 27′ 11.16″
6	23° 49′ 51″	14° 16′ 35″	19° 17′	37° 36′ 49.4″	54° 49′ 23.3″
7	44° 33′ 17″	32° 9′ 17″	32° 41′	49° 20′ 16.4″	50° 19′ 16″
8	97° 13′ 4″	132° 14′ 12″	79° 13′ 38.2″	131° 43′ 50″	81° 58′ 53.3″
	a	*A*	*c*	*b*	*B*
9	77° 21′ 50″	83° 56′ 40″	78° 53′ 20″	28° 14′ 31.8″	28° 49′ 57.4″
			101° 6′ 40″	151° 45′ 28.7″	151° 10′ 2.6″
10	77° 21′ 50″	40° 40′ 40″	impossible;	why?	
	a	*B*	*c*	*b*	*A*
11	92° 47′ 32″	50° 2′ 1″	91° 47′ 40″	50°	92° 8′ 23″
12	2° 0′ 55″	12° 40′	2° 3′ 55.7″	0° 27′ 10.2″	77° 20′ 28.4″
13	20° 20′ 20″	38° 10′ 10″	25° 14′ 38.2″	15° 16′ 50.4″	54° 35′ 16.7″
14	54° 30′	35° 30′	59° 51′ 20.8″	30° 8′ 39.2″	70° 17′ 35″
	c	*A*	*a*	*b*	*B*
15	69° 25′ 11″	54° 54′ 42″	50°	56° 50′ 49.3″	63° 25′ 4″
16	112° 48′	56° 11′ 56″	50°	127° 4′ 30″	120° 3′ 50″
17	46° 40′ 12″	37° 46′ 9″	26° 27′ 24″	39° 57′ 41.5″	62° 0′ 4″
18	118° 40′ 1″	128° 0′ 4″	136° 15′ 32.3″	48° 23′ 38.4″	58° 27′ 4.3″
	A	*B*	*a*	*b*	*c*
19	63° 15′ 12″	135° 33′ 39″	49° 59′ 56″	143° 5′ 12″	120° 55′ 34.3″
20	116° 43′ 12″	116° 31′ 25″	120° 10′ 3″	119° 59′ 46″	75° 26′ 58″
21	46° 59′ 42″	57° 59′ 17″	36° 27′	43° 32′ 30″.	54° 20′
22	90°	88° 24′ 35″	90°	88° 24′ 35″	90°

23. Define a quadrantal triangle, and show how its solution may be reduced to that of the right triangle.

24. Solve the quadrantal triangle whose sides are:

$$a = 174^\circ\,12'49.1'',\quad b = 94^\circ\,8'20'',\quad c = 90^\circ.$$

25. Solve the quadrantal triangle in which

$$c = 90^\circ,\quad A = 110^\circ 47' 50'',\quad B = 135^\circ 35' 34.5''.$$

26. Given in a spherical triangle A, C, and $c = 90^\circ$; solve the triangle.

27. Given $A = 60^\circ$, $C = 90^\circ$, and $c = 90^\circ$; solve the triangle.

28. Given in a right spherical triangle, $A = 42^\circ\,24'\,9''$, $B = 9^\circ\,4'\,11''$; solve the triangle.

29. In a right triangle, given $a = 119^\circ\,11'$, $B = 126^\circ\,54'$; solve the triangle.

30. In a right triangle, given $c = 50^\circ$, $b = 44^\circ\,18'39''$; solve the triangle.

31. In a right triangle, given $A = 156^\circ\,20'30''$, $a = 65^\circ\,15'45''$; solve the triangle.

32. If the legs a and b of a right spherical triangle are equal, prove that $\cos a = \cot A = \sqrt{\cos c}$.

33. In a right triangle prove that $\cos^2 A \times \sin^2 c = \sin(c - a)\sin(c + a)$.

34. In a right triangle prove that $\tan a \cos c = \sin b \cot B$.

35. In a right triangle prove that

$$\sin^2 A = \cos^2 B + \sin^2 a \sin^2 B.$$

36. In a right triangle prove that

$$\sin(b + c) = 2\cos^2 \tfrac{1}{2} A \cos b \sin c.$$

37. In a right triangle prove that

$$\sin(c - b) = 2\sin^2 \tfrac{1}{2} A \cos b \sin c.$$

38. If, in a right triangle, p denote the arc of the great circle passing through the vertex of the right angle and perpendicular to the hypotenuse, m and n the segments of the hypotenuse made by this arc adjacent to the legs a and b respectively, prove that (i.) $\tan^2 a = \tan c \tan m$, (ii.) $\sin^2 p = \tan m \tan n$.

§ 49. SOLUTION OF THE ISOSCELES SPHERICAL TRIANGLE.

If an arc of a great circle is passed through the vertex of an isosceles spherical triangle and the middle point of its base, the triangle will be divided into two symmetrical right triangles. In this way the solution of an isosceles spherical triangle may be reduced to that of a right spherical triangle.

In a similar manner the solution of a regular spherical polygon may be reduced to that of a right spherical triangle. Arcs of great circles, passed through the centre of the polygon and its vertices, divide it into a series of equal isosceles triangles; and each one of these may be divided into two equal right triangles.

EXERCISE XXIII.

1. In an isosceles spherical triangle, given the base b and the side a; find A the angle at the base, B the angle at the vertex, and h the altitude.

2. In an equilateral spherical triangle, given the side a; find the angle A.

3. Given the side a of a regular spherical polygon of n sides; find the angle A of the polygon, the distance R from the centre of the polygon to one of its vertices, and the distance r from the centre to the middle point of one of its sides.

4. Compute the dihedral angles made by the faces of the five regular polyhedrons.

5. A spherical square is a spherical quadrilateral which has equal sides and equal angles. Its two diagonals divide it into four equal right triangles. Find the angle A of the square, having given the side a.

CHAPTER VI.

THE OBLIQUE SPHERICAL TRIANGLE.

§ 51. FUNDAMENTAL FORMULAS.

Let ABC (Fig. 41) be an oblique spherical triangle, a, b, c its three sides, A, B, C the angles opposite to them, respectively.

Through C draw an arc CD of a great circle, perpendicular to the side AB, meeting AB at D. For brevity let $CD = p$, $AD = m$, $BD = n$, $\angle ACD = x$, $\angle BCD = y$.

1. By § 46 [38], in the right triangles BDC and ADC,

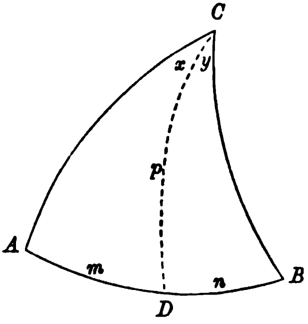

Fig. 41.

$$\sin p = \sin a \sin B,$$

and $$\sin p = \sin b \sin A.$$

$$\begin{array}{ll} \text{Therefore,} & \left.\begin{array}{l} \mathbf{\sin a \sin B = \sin b \sin A} \\ \mathbf{\sin a \sin C = \sin c \sin A} \\ \mathbf{\sin b \sin C = \sin c \sin B} \end{array}\right\}. \qquad [43] \\ \text{similarly,} & \\ \text{and} & \end{array}$$

These equations may of course also be written in the form of proportions; as, for example,

$$\sin a : \sin b = \sin A : \sin B.$$

In Fig. 41 the arc of the great circle CD cuts the side AB within the triangle. In case it cut AB produced without the triangle, $\sin(180° - A)$, $\sin(180° - B)$, or $\sin(180° - C)$, would be employed in the above proof instead of $\sin A$, $\sin B$, or $\sin C$. These sines, however, are equal to $\sin A$, $\sin B$, and $\sin C$, respectively, so that the Formulas [43] hold true in all cases.

2. In the right triangle BDC, by § 46 [37],

$$\cos a = \cos p \cos n = \cos p \cos(c - m),$$

or (§ 32) $\cos a = \cos p \cos c \cos m + \cos p \sin c \sin m.$

Now, $\cos p \cos m = \cos b;$ (§ 46 [37])

whence, $\cos p = \cos b \sec m,$

and $\cos p \sin m = \cos b \tan m$

$= \cos b \tan b \cos A$ (§ 46 [39])

$= \sin b \cos A.$

Substituting these values of $\cos p \cos m$ and $\cos p \sin m$ in the value of $\cos a$, we obtain

$$\left.\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A \\ \text{and similarly, } \cos b &= \cos a \cos c + \sin a \sin c \cos B \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C \end{aligned}\right\}. \qquad [44]$$

3. In the right triangle ADC, by § 46 [41],

$$\cos A = \cos p \sin x = \cos p \sin(C - y),$$

or (§ 32) $\cos A = \cos p \sin C \cos y - \cos p \cos C \sin y.$

Now, $\cos p \sin y = \cos B;$ (§ 46 [40])

whence, $\cos p = \cos B \csc y,$

and $\cos p \cos y = \cos B \cot y$

$= \cos B \tan B \cos a$ (§ 46 [42])

$= \sin B \cos a.$

Substituting these values of $\cos p \sin y$ and $\cos p \cos y$ in the value of $\cos A$, we obtain

$$\left.\begin{aligned} \cos A &= -\cos B \cos C + \sin B \sin C \cos a \\ \text{and similarly, } \cos B &= -\cos A \cos C + \sin A \sin C \cos b \\ \cos C &= -\cos A \cos B + \sin A \sin B \cos c \end{aligned}\right\}. \qquad [45]$$

Formulas [44] and [45] are also universally true; for the same equations are obtained when the arc CD cuts the side AB without the triangle.

EXERCISE XXIV.

1. What do Formulas [43] become if $A = 90°$? if $B = 90°$? if $C = 90°$? if $a = 90°$? if $A = B = 90°$? if $a = b = 90°$?

2. What does the first of [44] become if $A = 0°$? if $A = 90°$? if $A = 180°$?

3. From Formulas [44] deduce Formulas [45], by means of the relations between polar triangles (§ 45).

§ 52. FORMULAS FOR THE HALF ANGLES AND SIDES.

From the first equation of [44],

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c};$$

whence,

$$\begin{aligned} 1 - \cos A &= \frac{\sin b \sin c + \cos b \cos c - \cos a}{\sin b \sin c} \\ &= \frac{\cos (b - c) - \cos a}{\sin b \sin c}, \end{aligned}$$

$$\begin{aligned} 1 + \cos A &= \frac{\sin b \sin c - \cos b \cos c + \cos a}{\sin b \sin c} \\ &= \frac{\cos a - \cos (b + c)}{\sin b \sin c}; \end{aligned}$$

or, by § 34 [16] and [17], and § 35 [23],

$$\sin^2 \tfrac{1}{2} A = \sin \tfrac{1}{2} (a + b - c) \sin \tfrac{1}{2} (a - b + c) \csc b \csc c,$$

$$\cos^2 \tfrac{1}{2} A = \sin \tfrac{1}{2} (a + b + c) \sin \tfrac{1}{2} (b + c - a) \csc b \csc c.$$

Now, let $\tfrac{1}{2}(a + b + c) = s$;

whence,

$$\begin{aligned} \tfrac{1}{2}(b + c - a) &= s - a, \\ \tfrac{1}{2}(a - b + c) &= s - b, \\ \tfrac{1}{2}(a + b - c) &= s - c. \end{aligned}$$

Then, by substitution and extraction of the square root,

$$\left.\begin{aligned}\sin\tfrac{1}{2}A &= \sqrt{\sin(s-b)\sin(s-c)\csc b\csc c}\\ \cos\tfrac{1}{2}A &= \sqrt{\sin s\sin(s-a)\csc b\csc c}\\ \tan\tfrac{1}{2}A &= \sqrt{\csc s\csc(s-a)\sin(s-b)\sin(s-c)}\end{aligned}\right\}. \quad [46]$$

In like manner it may be shown that

$$\begin{aligned}\sin\tfrac{1}{2}B &= \sqrt{\sin(s-a)\sin(s-c)\csc a\csc c},\\ \cos\tfrac{1}{2}B &= \sqrt{\sin s\sin(s-b)\csc a\csc c},\\ \tan\tfrac{1}{2}B &= \sqrt{\csc s\csc(s-b)\sin(s-a)\sin(s-c)}.\end{aligned}$$

$$\begin{aligned}\sin\tfrac{1}{2}C &= \sqrt{\sin(s-a)\sin(s-b)\csc a\csc b},\\ \cos\tfrac{1}{2}C &= \sqrt{\sin s\sin(s-c)\csc a\csc b},\\ \tan\tfrac{1}{2}C &= \sqrt{\csc s\csc(s-c)\sin(s-a)\sin(s-b)}.\end{aligned}$$

Again, from the first equation of [45],

$$\cos a = \frac{\cos B\cos C + \cos A}{\sin B\sin C};$$

whence,

$$1-\cos a = \frac{\sin B\sin C - \cos B\cos C - \cos A}{\sin B\sin C},$$

$$1+\cos a = \frac{\sin B\sin C + \cos B\cos C + \cos A}{\sin B\sin C}.$$

If we place $\frac{1}{2}(A+B+C)=S$, and proceed in the same manner as before, we obtain the following results:

$$\left.\begin{aligned}\sin\tfrac{1}{2}a &= \sqrt{-\cos S\cos(S-A)\csc B\csc C}\\ \cos\tfrac{1}{2}a &= \sqrt{\cos(S-B)\cos(S-C)\csc B\csc C}\\ \tan\tfrac{1}{2}a &= \sqrt{-\cos S\cos(S-A)\sec(S-B)\sec(S-C)}\end{aligned}\right\}. \quad [47]$$

And, in like manner,

$$\sin\tfrac{1}{2}b=\sqrt{-\cos S\cos(S-B)\csc A\csc C},$$
$$\cos\tfrac{1}{2}b=\sqrt{\cos(S-A)\cos(S-C)\csc A\csc C},$$
$$\tan\tfrac{1}{2}b=\sqrt{-\cos S\cos(S-B)\sec(S-A)\sec(S-C)}.$$

$$\sin\tfrac{1}{2}c=\sqrt{-\cos S\cos(S-C)\csc A\csc B},$$
$$\cos\tfrac{1}{2}c=\sqrt{\cos(S-A)\cos(S-B)\csc A\csc B},$$
$$\tan\tfrac{1}{2}c=\sqrt{-\cos S\cos(S-C)\sec(S-A)\sec(S-B)}.$$

§ 53. GAUSS'S EQUATIONS AND NAPIER'S ANALOGIES.

By § 31 [5],

$$\cos\tfrac{1}{2}(A+B)=\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B-\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B;$$

or, by substituting for $\cos\frac{1}{2}A$, $\cos\frac{1}{2}B$, $\sin\frac{1}{2}A$, $\sin\frac{1}{2}B$, their values given in § 52, and reducing,

$$\begin{aligned}\cos\tfrac{1}{2}(A+B)&=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}\times\sqrt{\frac{\sin s\sin(s-b)}{\sin a\sin c}}\\&\quad-\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}\times\sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin a\sin c}}\\&=\frac{\sin s-\sin(s-c)}{\sin c}\times\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}.\end{aligned}$$

This value, by applying §§ 33 [12], 35 [21], and observing that the quantity under the radical is equal to $\sin\frac{1}{2}C$, becomes

$$\cos\tfrac{1}{2}(A+B)=\frac{2\sin\frac{1}{2}c\cos(s-\frac{1}{2}c)}{2\sin\frac{1}{2}c\cos\frac{1}{2}c}\sin\tfrac{1}{2}C,$$

which, by cancelling common factors, multiplying by $\cos\frac{1}{2}c$, and observing that $s-\frac{1}{2}c=\frac{1}{2}(a+b)$, reduces to the form

$$\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c=\cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C.$$

By proceeding in like manner with the values of

$$\sin\tfrac{1}{2}(A+B),\quad \cos\tfrac{1}{2}(A-B),\quad \text{and } \sin\tfrac{1}{2}(A-B),$$

three analogous equations are obtained.

The four equations,

$$\left.\begin{aligned}
\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c &= \cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C\\
\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c &= \cos\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C\\
\cos\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c &= \sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C\\
\sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c &= \sin\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C
\end{aligned}\right\} \qquad [48]$$

are called Gauss's Equations.

By dividing the second of Gauss's Equations by the first, the fourth by the third, the third by the first, and the fourth by the second, we obtain

$$\left.\begin{aligned}
\tan\tfrac{1}{2}(A+B) &= \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C\\
\tan\tfrac{1}{2}(A-B) &= \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C\\
\tan\tfrac{1}{2}(a+b) &= \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c\\
\tan\tfrac{1}{2}(a-b) &= \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c
\end{aligned}\right\}. \qquad [49]$$

These equations are called Napier's Analogies.

In the first equation the factors $\cos\frac{1}{2}(a-b)$ and $\cot\frac{1}{2}C$ are always positive; therefore, $\tan\frac{1}{2}(A+B)$ and $\cos\frac{1}{2}(a+b)$ must always have like signs. Hence, if $a+b<180°$, and therefore $\cos\frac{1}{2}(a+b)>0$, then, also, $\tan\frac{1}{2}(A+B)>0$, and therefore $A+B<180°$. Similarly, it follows that if $a+b>180°$, then, also, $A+B>180°$. If $a+b=180°$, and therefore $\cos\frac{1}{2}(a+b)=0$, then $\tan\frac{1}{2}(A+B)=\infty$; whence $\frac{1}{2}(A+B)=90°$, and $A+B=180°$.

Conversely, it may be shown from the third equation, that $a+b$ is less than, greater than, or equal to 180°, according as $A+B$ is less than, greater than, or equal to 180°.

§ 54. Case I.

Given two sides, a and b, and the included angle C.

The angles A and B may be found by the first two of Napier's Analogies; viz.:

$$\tan\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C.$$

$$\tan\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C.$$

After A and B have been found, the side c may be found by [43]; but it is better to use for this purpose Gauss's Equations, because they involve functions of the same angles that occur in working Napier's Analogies. Any one of the equations may be used; for example, from the first we have

$$\cos\tfrac{1}{2}c=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(A+B)}\sin\tfrac{1}{2}C.$$

Example. $a=73°58'54''$, $b=38°45'\ 0''$, $C=46°33'41''$, therefore, $\tfrac{1}{2}(a-b)=17°36'57''$, $\tfrac{1}{2}(a+b)=56°21'57''$, $\tfrac{1}{2}C\ =23°16'50.5''$

$$\begin{array}{ll|ll}
\log\cos\frac{1}{2}(a-b) & =9.97914 & \log\sin\frac{1}{2}(a-b) & =9.48092\\
\log\sec\frac{1}{2}(a+b) & =0.25658 & \log\csc\frac{1}{2}(a+b) & =0.07956\\
\log\cot\frac{1}{2}C & =0.36626 & \log\cot\frac{1}{2}C & =0.36626\\
\hline
\log\tan\frac{1}{2}(A+B) & =0.60198 & \log\tan\frac{1}{2}(A-B) & =9.92674\\
\hline
\log\sec\frac{1}{2}(A+B) & =0.61515 & \frac{1}{2}(A+B) & =\ 75°57'40.7''\\
\log\cos\frac{1}{2}(a+b) & =9.74342 & \frac{1}{2}(A-B) & =\ 40°11'25.6''\\
\log\sin\frac{1}{2}C & =9.59686 & A & =116°\ 9'\ 6.3''\\
\hline
\log\cos\frac{1}{2}c & =9.95543 & B & =\ 35°46'15.1''\\
\frac{1}{2}c & =25°\ 31' & c & =\ 51°\ 2'
\end{array}$$

If the side c only is desired, it may be found from [44], without previously computing A and B. But the Formulas [44] are not adapted to logarithmic work. Instead of changing them to forms suitable for logarithms, we may use the following method, which leads to the same results, and has the advantage that, in applying it, nothing has to be remembered except Napier's Rules:

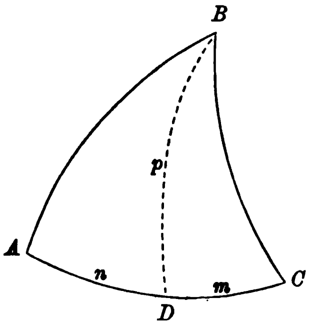

Fig. 42.

Make the triangle (Fig. 42), as in §51, equal to the sum (or the difference) of two right triangles. For this purpose, through B (or A, but *not* C) draw an arc of a great circle perpendicular to AC, cutting AC at D. Let $BD = p$, $CD = m$, $AD = n$; and mark with crosses the given parts.

By Rule I.,

$$\cos C = \tan m \cot a,$$

whence $\tan m = \tan a \cos C$.

By Rule II.,

$$\cos a = \cos m \cos p, \text{ whence } \cos p = \cos a \sec m.$$
$$\cos c = \cos n \cos p, \text{ whence } \cos p = \cos c \sec n.$$

Therefore, $\cos c \sec n = \cos a \sec m$;

or, since $n = b - m$, $\cos c = \cos a \sec m \cos (b - m)$.

It is evident that c may be computed, with the aid of logarithms, from the two equations

$$\tan m = \tan a \cos C,$$
$$\cos c = \cos a \sec m \cos (b - m).$$

Example. Given $a = 97° 30' 20''$, $b = 55° 12' 10''$, $C = 39° 58'$; find c.

$$\begin{array}{rl|rl}
\log \tan a & = 0.88025 (n) & \log \cos a & = 9.11602 (n) \\
\log \cos C & = 9.88447 & \log \cos (b - m) & = 9.85286 \\
\hline
\log \tan m & = 0.76472 (n) & \log \sec m & = 0.77103 (n) \\
\cline{3-4}
m & = \quad 99° 45' 14'' & \log \cos c & = 9.73991 \\
b - m & = -44° 33' \; 4'' & c & = 56° 40' 20''
\end{array}$$

Exercise XXV.

1. Write formulas for finding, by Napier's Rules, the side a when b, c, and A are given, and for finding the side b when a, c, and B are given.

2. Given $a = 88° 12' 20''$, $b = 124° 7' 17''$, $C = 50° 2' 1''$; find $A = 63° 15' 11''$, $B = 132° 17' 59''$, $c = 59° 4' 18''$.

3. Given $a = 120° 55' 35''$, $b = 88° 12' 20''$, $C = 47° 42' 1''$; find $A = 129° 58' 3''$, $B = 63° 15' 9''$, $c = 55° 52' 40''$.

4. Given $b = 63° 15' 12''$, $c = 47° 42' 1''$, $A = 59° 4' 25''$; find $B = 88° 12' 24''$, $C = 55° 52' 42''$, $a = 50° 1' 40''$.

5. Given $b = 69° 25' 11''$, $c = 109° 46' 19''$, $A = 54° 54' 42''$; find $B = 56° 11' 57''$, $C = 123° 21' 12''$, $a = 67° 13'$.

§ 55. CASE II.

Given two angles, A and B, and the included side c.

The sides a and b may be found by the third or fourth of Napier's Analogies,

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c,$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c,$$

and then the angle C may be found by one of Gauss's equations; as, for instance, the second, which gives

$$\cos \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(A+B)}{\cos \frac{1}{2}(a-b)} \cos \tfrac{1}{2} c.$$

EXAMPLE.

$$\begin{array}{rl}
A = & 107° 47' \ 7'' \\
B = & 38° 58' 27'' \\
c = & 51° 41' 14''
\end{array}
\qquad
\begin{array}{rl}
\frac{1}{2}(A-B) = & 34° 24' 20'' \\
\frac{1}{2}(A+B) = & 73° 22' 47'' \\
\frac{1}{2}c = & 25° 50' 37''
\end{array}$$

$$\begin{array}{ll}
\log \cos \frac{1}{2}(A-B) & = 9.91648 \\
\log \sec \frac{1}{2}(A+B) & = 0.54359 \\
\log \tan \frac{1}{2} c & = 9.68517 \\ \hline
\log \tan \frac{1}{2}(a+b) & = 0.14524 \\ \hline
\log \sin \frac{1}{2}(A+B) & = 9.98146 \\
\log \sec \frac{1}{2}(a-b) & = 0.01703 \\
\log \cos \frac{1}{2} c & = 9.95423 \\ \hline
\log \cos \frac{1}{2} C & = 9.95272 \\
\frac{1}{2} C & = 26° 14' 52.5''
\end{array}
\qquad
\begin{array}{ll}
\log \sin \frac{1}{2}(A-B) & = 9.75208 \\
\log \csc \frac{1}{2}(A+B) & = 0.01854 \\
\log \tan \frac{1}{2} c & = 9.68517 \\ \hline
\log \tan \frac{1}{2}(a-b) & = 9.45579 \\ \hline
\frac{1}{2}(a+b) & = 54° 24' 24.4'' \\
\frac{1}{2}(a-b) & = 15° 56' 25.6''
\end{array}
\quad
\left\{ \begin{array}{l}
a = 70° 20' 50'' \\
b = 38° 27' 59'' \\
C = 52° 29' 45''
\end{array} \right.$$

If the angle C alone is wanted, the best way is to decompose the triangle into two right triangles, and then apply Napier's Rules, as in Case I., when the side c alone is desired.

Let (Fig. 43) $\angle ABD = x$, $\angle CBD = y$, $BD = p$; then,

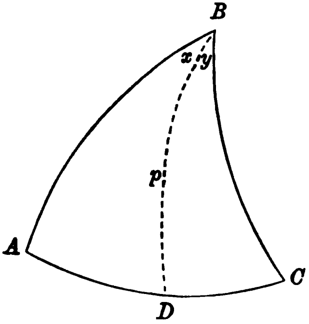

Fig. 43.

By Rule I.,

$$\cos c = \cot x \cot A,$$

whence $\cot x = \tan A \cos c$.

By Rule II.,

$$\cos A = \cos p \sin x,$$

whence $\cos p = \cos A \csc x$.

$$\cos C = \cos p \sin y,$$

whence $\cos p = \cos C \csc y$.

Therefore,

$$\cos C = \cos A \csc x \sin y = \cos A \csc x \sin (B - x).$$

It is clear that C may be computed from the equations

$$\cot x = \tan A \cos c,$$
$$\cos C = \cos A \csc x \sin (B - x).$$

EXAMPLE. Given $A = 35° 46' 15''$, $B = 115° 9' 7''$, $c = 51° 2'$; find C.

$$\begin{array}{rl|rl}
\log \tan A & = 9.85760 & \log \cos A & = 9.90992 \\
\log \cos c & = 9.79856 & \log \sin (B - x) & = 9.88122 \\
\hline
\log \cot x & = 9.65616 & \log \csc x & = 0.04055 \\
x & = 65° 37' 35'' & \log \cos C & = 9.83099 \\
B - x & = 49° 31' 32'' & C & = 47° 20' 30''
\end{array}$$

EXERCISE XXVI.

1. What are the formulas for computing A when B, C, and a are given; and for computing B when A, C, and b are given?

2. Given $A = 26° 58' 46''$, $B = 39° 45' 10''$, $c = 154° 46' 48''$; find $a = 37° 14' 10''$, $b = 121° 28' 10''$, $C = 161° 22' 11''$.

3. Given $A = 128^\circ\, 41'\, 49''$, $B = 107^\circ\, 33'\, 20''$, $c = 124^\circ\, 12$ $31''$; find $a = 125^\circ\, 41'\, 44''$, $b = 82^\circ\, 47'\, 34''$, $C = 127^\circ\, 22'$.

4. Given $B = 153^\circ\, 17'\, 6''$, $C = 78^\circ\, 43'\, 36''$, $a = 86^\circ\, 15'\, 15''$; find $b = 152^\circ\, 43'\, 51''$, $c = 88^\circ\, 12'\, 21''$, $A = 78^\circ\, 15'\, 48''$.

5. Given $A = 125^\circ\, 41'\, 44''$, $C = 82^\circ\, 47'\, 35''$, $b = 52^\circ\, 37'\, 57''$; find $a = 128^\circ\, 31'\, 46''$, $c = 107^\circ\, 33'\, 20''$, $B = 55^\circ\, 47'\, 40''$.

§ 56. Case III.

Given two sides a and b, and the angle A opposite to one of them.

The angle B is found from [43], whence we have

$$\sin B = \sin A \sin b \csc a.$$

When B has been found, C and c may be found from the fourth and the second of Napier's Analogies, from which we obtain

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a - b),$$

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)} \tan \tfrac{1}{2}(A - B).$$

The third and the first of Napier's Analogies may also be used for the same purpose.

Note 1. Since B is determined from its sine, the problem in general has two solutions; and, moreover, in case $\sin B > 1$, the problem is impossible. By geometric construction it may be shown, as in the corresponding case in Plane Trigonometry, under what conditions the problem really has two solutions, one solution, and no solution. But in practical applications a general knowledge of the shape of the triangle is known beforehand; so that it is easy to see, without special investigation, which solution (if any) corresponds to the circumstances of the question.

It can be shown that in general there are two solutions, when

$$A < 90^\circ,\ a + b < 180^\circ,\ \text{and } a < b,$$

or, when

$$A > 90^\circ,\ a + b > 180^\circ,\ \text{and } a > b.$$

Note 2. The side c or the angle C may be computed, without first finding B, by means of the formulas

$$\tan m = \cos A \tan b, \text{ and } \cos(c-m) = \cos a \times \sec b \times \cos m,$$
$$\cot x = \tan A \cos b, \text{ and } \cos(C-x) = \cot a \times \tan b \times \cos x.$$

These formulas may be obtained by resolution of the triangle into right triangles, and applying Napier's Rules; m is equal to that part of the side c included between the foot of the perpendicular from C and the vertex A, and x is equal to the corresponding portion of the angle C.

Example. Given $a = 57^\circ 36'$, $b = 31^\circ 12'$, $A = 104^\circ 25' 30''$.

In this case $A > 90^\circ$,
and $a + b < 180^\circ$;
therefore, $A + B < 180^\circ$;
hence, $B < 90^\circ$,
and only one solution.

$$\begin{aligned} \log\sin A &= 9.98609 \\ \log\sin b &= 9.71435 \\ \log\csc a &= 0.07349 \\ \hline \log\sin B &= 9.77393 \\ B &= 36^\circ 27' 20'' \end{aligned}$$

$$\begin{aligned} a + b &= 88^\circ 50' \\ a - b &= 26^\circ 26' \\ A + B &= 140^\circ 51' 53'' \\ A - B &= 67^\circ 59' \ 7'' \\ \log\sin \tfrac{1}{2}(A+B) &= 9.97416 \\ \log\csc \tfrac{1}{2}(A-B) &= 0.25252 \\ \log\tan \tfrac{1}{2}(a-b) &= 9.37080 \\ \hline \log\tan \tfrac{1}{2}c &= 9.59748 \\ \tfrac{1}{2}c &= 21^\circ 35' 38'' \\ c &= 43^\circ 11' 16'' \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(a+b) &= 44^\circ 25' \\ \tfrac{1}{2}(a-b) &= 13^\circ 13' \\ \tfrac{1}{2}(A+B) &= 70^\circ 26' 25'' \\ \tfrac{1}{2}(A-B) &= 33^\circ 59' \ 5'' \\ \log\sin \tfrac{1}{2}(a+b) &= 9.84502 \\ \log\csc \tfrac{1}{2}(a-b) &= 0.64086 \\ \log\tan \tfrac{1}{2}(A-B) &= 9.82873 \\ \hline \log\cot \tfrac{1}{2}C &= 0.31461 \\ \tfrac{1}{2}C &= 25^\circ 51' 15'' \\ C &= 51^\circ 42' 30'' \end{aligned}$$

Exercise XXVII.

1. Given $a = 73^\circ 49' 38''$, $b = 120^\circ 53' 35''$, $A = 88^\circ 52' 42''$; find $B = 116^\circ 42' 30''$, $c = 120^\circ 57' 27''$, $C = 116^\circ 47' 4''$.

2. Given $a = 150^\circ 57' 5''$, $b = 134^\circ 15' 54''$, $A = 144^\circ 22' 42''$; find $B_1 = 120^\circ 47' 45''$, $c_1 = 55^\circ 42' 8''$, $C_1 = 97^\circ 42' 55.4''$;
$B_2 = 59^\circ 12' 15''$, $c_2 = 23^\circ 57' 17.4''$, $C_2 = 29^\circ \ 8' 39''$.

3. Given $a = 79^\circ 0' 54.5''$, $b = 82^\circ 17' 4''$, $A = 82^\circ 9' 25.8''$; find $B = 90^\circ$, $c = 45^\circ 12' 19''$, $C = 45^\circ 44'$.

4. Given $a = 30^\circ 52' 36.6''$, $b = 31^\circ 9' 16''$, $A = 87^\circ 34' 12''$; show that the triangle is impossible.

§ 57. CASE IV.

Given two angles A and B, and the side a opposite to one of them.

The side b is found from [43], whence

$$\sin b = \sin a \sin B \csc A.$$

The values of c and C may then be found by means of Napier's Analogies, the fourth and second of which give

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a-b),$$

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(A-B).$$

NOTE 1. In this case, also, an unknown part is found from its sine; and it may be shown that, under certain conditions, the problem is impossible, or that it admits of two solutions. In practice, the ambiguity is usually removed by the circumstances of the question. If $\sin b > 1$ the problem of course is impossible; and it may be shown that there are two solutions, when

$$a < 90°, \quad A + B < 180°, \quad \text{and } A < B,$$

or

$$a > 90°, \quad A + B > 180°, \quad \text{and } A > B.$$

NOTE 2. By proceeding as indicated in Case III., Note 2, formulas for computing c or C, independent of the side b, may be found; viz.:

$$\tan m = \tan a \cos B, \text{ and } \sin(c - m) = \cot A \tan B \sin m,$$
$$\cot x = \cos a \tan B, \text{ and } \sin(C - x) = \cos A \sec B \sin x.$$

In these formulas $m = BD$, $x = \angle BCD$, D being the foot of the perpendicular from the vertex C.

EXERCISE XXVIII.

1. Given $A = 110° 10'$, $B = 133° 18'$, $a = 147° 5' 32''$; find $b = 155° 5' 18''$, $c = 33° 1' 36''$, $C = 70° 20' 40''$.

2. Given $A = 113° 39' 21''$, $B = 123° 40' 18''$, $a = 65° 39' 46''$; find $b = 124° 7' 20''$, $c = 159° 53' 2''$, $C = 159° 43' 35''$.

3. Given $A = 100° 2' 11.3''$, $B = 98° 30' 28''$, $a = 95° 20' 38.7''$; find $b = 90°$, $c = 147° 41' 43''$, $C = 148° 5' 33''$.

4. Given $A = 24° 33' 9''$, $B = 38° 0' 12''$, $a = 65° 20' 13''$; show that the triangle is impossible.

§ 58. CASE V.

Given the three sides, a, b, and c.

The angles are computed by means of Formulas [46], and the corresponding formulas for the angles B and C.

The formulas for the tangent are in general to be preferred.

If we multiply the equation

$$\tan \tfrac{1}{2} A = \sqrt{\csc s \csc (s-a) \sin (s-b) \sin (s-c)}$$

by the equation
$$1 = \frac{\sin (s-a)}{\sin (s-a)},$$

and put

$$\sqrt{\csc s \sin (s-a) \sin (s-b) \sin (s-c)} = \tan r,$$

and also make analogous changes in the equations for $\tan \tfrac{1}{2} B$ and $\tan \tfrac{1}{2} C$, we obtain

$$\begin{aligned} \tan \tfrac{1}{2} A &= \tan r \csc (s-a), \\ \tan \tfrac{1}{2} B &= \tan r \csc (s-b), \\ \tan \tfrac{1}{2} C &= \tan r \csc (s-c), \end{aligned}$$

which are the most convenient formulas to employ when all three angles have to be computed.

EXAMPLE 1.

$$\begin{aligned} a &= 50° 54' 32'' \\ b &= 37° 47' 18'' \\ c &= 74° 51' 50'' \\ \hline 2s &= 163° 33' 40'' \\ s &= 81° 46' 50'' \\ s-a &= 30° 52' 18'' \\ s-b &= 43° 59' 32'' \\ s-c &= 6° 55' 0'' \end{aligned}$$

$$\begin{aligned} \log \csc s &= 0.00448 \\ \log \csc (s-a) &= 0.28978 \\ \log \sin (s-b) &= 9.84171 \\ \log \sin (s-c) &= 9.08072 \\ 2 &\overline{)19.21669} \\ \log \tan \tfrac{1}{2} A &= 9.60835 \\ \tfrac{1}{2} A &= 22° 5' 20'' \\ A &= 44° 10' 40'' \end{aligned}$$

EXAMPLE 2.

$$\begin{aligned} a &= 124^\circ\, 12'\, 31'' \\ b &= 54^\circ\, 18'\, 10'' \\ c &= 97^\circ\, 12'\, 25'' \\ \hline 2s &= 275^\circ\, 43'\, 12'' \end{aligned}$$

$$\begin{aligned} \log\sin(s-a) &= 9.37293 \\ \log\sin(s-b) &= 9.99725 \\ \log\sin(s-c) &= 9.81390 \\ \log\csc s &= 0.17331 \\ \hline \log\tan^2 r &= 9.35739 \\ \log\tan r &= 9.67870 \end{aligned}$$

$$\begin{aligned} s &= 137^\circ\, 51'\, 36'' \\ s-a &= 13^\circ\, 39'\, 5'' \\ s-b &= 83^\circ\, 33'\, 20'' \\ s-c &= 40^\circ\, 39'\, 11'' \end{aligned}$$

$$\begin{aligned} \log\tan\tfrac{1}{2}A &= 0.30577 \\ \log\tan\tfrac{1}{2}B &= 9.68145 \\ \log\tan\tfrac{1}{2}C &= 9.86480 \\ \hline \tfrac{1}{2}A &= 63^\circ\, 41'\, 3.8'' \\ \tfrac{1}{2}B &= 25^\circ\, 39'\, 5.6'' \\ \tfrac{1}{2}C &= 36^\circ\, 13'\, 20.1'' \\ \hline A &= 127^\circ\, 22'\, 7'' \\ B &= 51^\circ\, 18'\, 11'' \\ C &= 72^\circ\, 26'\, 40'' \end{aligned}$$

EXERCISE XXIX.

1. Given $a = 120^\circ\, 55'\, 35''$, $b = 59^\circ\, 4'\, 25''$, $c = 106^\circ\, 10'\, 22''$; find $A = 116^\circ\, 44'\, 49''$, $B = 63^\circ\, 15'\, 14''$, $C = 91^\circ\, 7'\, 21''$.

2. Given $a = 50^\circ\, 12'\, 4''$, $b = 116^\circ\, 44'\, 48''$, $c = 129^\circ\, 11'\, 42''$; find $A = 59^\circ\, 4'\, 28''$, $B = 94^\circ\, 23'\, 12''$, $C = 120^\circ\, 4'\, 52''$.

3. Given $a = 131^\circ\, 35'\, 4''$, $b = 108^\circ\, 30'\, 14''$, $c = 84^\circ\, 46'\, 34''$; find $A = 132^\circ\, 14'\, 21''$, $B = 110^\circ\, 10'\, 40''$, $C = 99^\circ\, 42'\, 24''$.

4. Given $a = 20^\circ\, 16'\, 38''$, $b = 56^\circ\, 19'\, 40''$, $c = 66^\circ\, 20'\, 44''$; find $A = 20^\circ\, 9'\, 54''$, $B = 55^\circ\, 52'\, 31''$, $C = 114^\circ\, 20'\, 17''$.

§ 59. CASE VI.

Given the three angles, A, B, and C.

The sides are computed by means of Formulas [47], and the corresponding formulas for the angles B and C. The formulas for the tangents are in general to be preferred.

If we multiply the equation

$$\tan\tfrac{1}{2}a = \sqrt{-\cos S\cos(S-A)\sec(S-B)\sec(S-C)}$$

by the equation $$1 = \frac{\sec(S-A)}{\sec(S-A)},$$
and put

$$\sqrt{-\cos S\sec(S-A)\sec(S-B)\sec(S-C)} = \tan R,$$

and also make analogous changes in the equations for $\tan\frac{1}{2}b$ and $\tan\frac{1}{2}c$, we obtain

$$\begin{aligned}\tan\tfrac{1}{2}a &= \tan R\cos(S-A),\\ \tan\tfrac{1}{2}b &= \tan R\cos(S-B),\\ \tan\tfrac{1}{2}c &= \tan R\cos(S-C),\end{aligned}$$

which are the most convenient formulas to use in case all three angles have to be computed.

EXAMPLE I.

$$\begin{aligned}A &= 220^\circ\\ B &= 130^\circ\\ C &= 150^\circ\\ \hline 2S &= 500^\circ\\ S &= 250^\circ\\ S-A &= 30^\circ\\ S-B &= 120^\circ\\ S-C &= 100^\circ\end{aligned}$$

$$\begin{aligned}\log\cos S &= 9.53405\ (n)\\ \log\cos(S-A) &= 9.93753\\ \log\sec(S-B) &= 0.30103\ (n)\\ \log\sec(S-C) &= 0.76033\ (n)\\ 2&\overline{)0.53294}\\ \log\tan\tfrac{1}{2}a &= 0.26647\\ \tfrac{1}{2}a &= 61^\circ 34' 6''\\ a &= 123^\circ 8' 12''\end{aligned}$$

NOTE. Here the effect, as regards algebraic sign, of three negative factors, is cancelled by the negative sign belonging to the whole fraction.

EXAMPLE II.

$$\begin{aligned}A &= 20^\circ 9' 56''\\ B &= 55^\circ 52' 32''\\ C &= 114^\circ 20' 14''\\ \hline 2S &= 190^\circ 22' 42''\end{aligned}$$

$$\begin{aligned}S &= 95^\circ 11' 21''\\ S-A &= 75^\circ 1' 25''\\ S-B &= 39^\circ 18' 49''\\ S-C &= -19^\circ 8' 53''\end{aligned}$$

$$\begin{aligned}\log\cos S &= 8.95638\ (n)\\ \log\sec(S-A) &= 0.58768\\ \log\sec(S-B) &= 0.11143\\ \log\sec(S-C) &= 0.02472\\ \hline \log\tan^2 R &= 9.68021\\ \log\tan R &= 9.84010\end{aligned}$$

$$\begin{aligned}\log\tan\tfrac{1}{2}a &= 9.25242\\ \log\tan\tfrac{1}{2}b &= 9.72867\\ \log\tan\tfrac{1}{2}c &= 9.81538\\ \hline \tfrac{1}{2}a &= 10^\circ 8' 18.9''\\ \tfrac{1}{2}b &= 28^\circ 9' 50.4''\\ \tfrac{1}{2}c &= 33^\circ 10' 21.3''\\ \hline a &= 20^\circ 16' 38''\\ b &= 56^\circ 19' 41''\\ c &= 66^\circ 20' 43''\end{aligned}$$

EXERCISE XXX.

1. Given $A = 130^\circ$, $B = 110^\circ$, $C = 80^\circ$;
find $a = 139^\circ 21' 22''$, $b = 126^\circ 57' 52''$, $c = 56^\circ 51' 48''$.

2. Given $A = 59^\circ 55' 10''$, $B = 85^\circ 36' 50''$, $C = 59^\circ 55' 10''$;
find $a = 51^\circ 17' 31''$, $b = 64^\circ 2' 47''$, $c = 51^\circ 17' 31''$.

3. Given $A = 102° 14' 12''$, $B = 54° 32' 24''$, $C = 89° 5' 46''$: find $a = 104° 25' 9''$, $b = 53° 49' 25''$, $c = 97° 44' 24''$.

4. Given $A = 4° 23' 35''$, $B = 8° 28' 20''$, $C = 172° 17' 56''$; find $a = 31° 9' 11''$, $b = 84° 18' 23''$, $c = 115° 9' 56''$.

§ 60. AREA OF A SPHERICAL TRIANGLE.

I. *When the three angles A, B, C, are given.*

Let $R =$ radius of sphere,
$E =$ the spherical excess $= A + B + C - 180°$,
$F =$ area of triangle;

then, by Solid Geometry,

$$\mathbf{F} = \frac{\mathbf{E}}{180°}\pi \mathbf{R}^2. \qquad [50]$$

II. *When the three sides a, b, c, are given.*

A formula for computing the area is deduced as follows:

From the first of [48],

$$\frac{\cos\frac{1}{2}(A+B)}{\cos(90° - \frac{1}{2}C)} = \frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c};$$

whence, by the Theory of Proportions,

$$\frac{\cos\frac{1}{2}(A+B) - \cos(90° - \frac{1}{2}C)}{\cos\frac{1}{2}(A+B) + \cos(90° - \frac{1}{2}C)} = \frac{\cos\frac{1}{2}(a+b) - \cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b) + \cos\frac{1}{2}c}. \qquad \text{(a)}$$

Now, in § 35, the division of [23] by [22] gives

$$\frac{\cos A - \cos B}{\cos A + \cos B} = -\tan\frac{1}{2}(A+B)\tan\frac{1}{2}(A-B), \qquad \text{(b)}$$

in which for A and B we may substitute any other two angular magnitudes, as for example, $\frac{1}{2}(A+B)$ and $(90 - \frac{1}{2}C)$, or $\frac{1}{2}(a+b)$ and $\frac{1}{2}c$.

If we use in place of A and B the values $\frac{1}{2}(A+B)$ and $(90° - \frac{1}{2}C)$, the first side of equation (b) becomes

$$\frac{\cos\frac{1}{2}(A+B) - \cos(90° - \frac{1}{2}C)}{\cos\frac{1}{2}(A+B) + \cos(90° - \frac{1}{2}C)};$$

and the second side becomes

$$-\tan\tfrac{1}{2}(\tfrac{1}{2}A+\tfrac{1}{2}B+90^\circ-\tfrac{1}{2}C)\tan\tfrac{1}{2}(\tfrac{1}{2}A+\tfrac{1}{2}B-90^\circ+\tfrac{1}{2}C);$$

or,

$$-\tan\tfrac{1}{4}(A+B-C+180^\circ)\tan\tfrac{1}{4}(A+B+C-180^\circ).$$

If we remember that $E=A+B+C-180^\circ$, and observe that

$$\begin{aligned}\tan\tfrac{1}{4}(A+B-C+180^\circ)&=\tan\tfrac{1}{4}(360^\circ-2C+A+B+C-180^\circ)\\&=\tan\tfrac{1}{4}(360^\circ-2C+E)\\&=\tan[90^\circ-\tfrac{1}{4}(2C-E)]\\&=\cot\tfrac{1}{4}(2C-E),\end{aligned}$$

it will be evident that equation (b) may be written

$$\frac{\cos\frac{1}{2}(A+B)-\cos(90^\circ-\frac{1}{2}C)}{\cos\frac{1}{2}(A+B)+\cos(90^\circ-\frac{1}{2}C)}=-\cot\tfrac{1}{4}(2C-E)\tan\tfrac{1}{4}E.\quad\text{(c)}$$

If we substitute, in equation (b), for A and B, the values $\frac{1}{2}(a+b)$ and $\frac{1}{2}c$, and also substitute s for $\frac{1}{2}(a+b+c)$ and $s-c$ for $\frac{1}{2}(a+b-c)$, equation (b) will become

$$\frac{\cos\frac{1}{2}(a+b)-\cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b)+\cos\frac{1}{2}c}=-\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-c).\qquad\text{(d)}$$

Comparing (a), (c), and (d), we obtain

$$\cot\tfrac{1}{4}(2C-E)\tan\tfrac{1}{4}E=\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-c).\qquad\text{(e)}$$

By beginning with the second equation of [48], and treating it in the same way, we obtain as the result,

$$\tan\tfrac{1}{4}(2C-E)\tan\tfrac{1}{4}E=\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b).\qquad\text{(f)}$$

By taking the product of (e) and (f), we obtain the elegant formula,

$$\tan^2\tfrac{1}{4}\mathbf{E}=\tan\tfrac{1}{2}\mathbf{s}\tan\tfrac{1}{2}(\mathbf{s}-\mathbf{a})\tan\tfrac{1}{2}(\mathbf{s}-\mathbf{b})\tan\tfrac{1}{2}(\mathbf{s}-\mathbf{c}),\qquad[51]$$

which is known as l'Huilier's Formula.

By means of it E may be computed from the three sides, and then the area of the triangle may be found by [50].

III. *In all other cases*, the area may be found by first solving the triangle so far as to obtain the angles or the sides, whichever may be more convenient, and then applying [50] or [51].

Example I.

$$\begin{aligned} A &= 102^\circ\,14'\,12'' \\ B &= 54^\circ\,32'\,24'' \\ C &= 89^\circ\,5'\,46'' \\ \hline & \ 245^\circ\,52'\,22'' \\ E &= 65^\circ\,52'\,22'' \\ &= 237142'' \\ 180^\circ &= 648000'' \end{aligned}$$

$\log E = 5.37501$

*$\log\dfrac{\pi}{648000} = 4.68557 - 10$

$\ 0.06058$

$F = 1.1497\,R^2$

Example II.

$$\begin{aligned} a &= 133^\circ\,26'\,19'' \\ b &= 64^\circ\,50'\,53'' \\ c &= 144^\circ\,13'\,45'' \\ \hline 2s &= 342^\circ\,30'\,57'' \\ \hline s &= 171^\circ\,15'\,28.5'' \\ s-a &= 37^\circ\,49'\,9.5'' \\ s-b &= 106^\circ\,24'\,35.5'' \\ s-c &= 27^\circ\,1'\,43.5'' \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}s &= 85^\circ\,37'\,44'' \\ \tfrac{1}{2}(s-a) &= 18^\circ\,54'\,35'' \\ \tfrac{1}{2}(s-b) &= 53^\circ\,12'\,18'' \\ \tfrac{1}{2}(s-c) &= 13^\circ\,30'\,52'' \\ \hline \log\tan\tfrac{1}{2}s &= 1.11669 \\ \log\tan\tfrac{1}{2}(s-a) &= 9.53474 \\ \log\tan\tfrac{1}{2}(s-b) &= 0.12612 \\ \log\tan\tfrac{1}{2}(s-c) &= 9.38083 \\ \hline \log\tan^2\tfrac{1}{4}E &= 0.15838 \\ \log\tan\tfrac{1}{4}E &= 0.07919 \\ \tfrac{1}{4}E &= 50^\circ\,11'\,43'' \\ E &= 200^\circ\,46'\,52'' \end{aligned}$$

EXERCISE XXXI.

1. Given $A = 84^\circ\,20'\,19''$, $B = 27^\circ\,22'\,40''$, $C = 75^\circ\,33'$; find $E = 26159''$, $F = 0.12685\,R^2$.

2. Given $a = 69^\circ\,15'\,6''$, $b = 120^\circ\,42'\,47''$, $c = 159^\circ\,18'\,33''$; find $E = 216^\circ\,40'\,18''$.

3. Given $a = 33^\circ\,1'\,45''$, $b = 155^\circ\,5'\,18''$, $C = 110^\circ\,10'$; find $E = 133^\circ\,48'\,53''$.

4. Find the spherical excess of a triangle on the earth's surface (regarded as spherical), if each side of the triangle is equal to 1°.

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* See Wentworth & Hill's Tables, page 20.

CHAPTER VII.

APPLICATIONS OF SPHERICAL TRIGONOMETRY.

§ 61. PROBLEM.

To reduce an angle measured in space to the horizon.

Let O (Fig. 44) be the position of the observer on the ground, $AOB = h$ the angle measured in space (for example, the angle between the tops of two church spires), OA' and OB' the projections of the sides of the angle upon the horizontal plane HR, $AOA' = m$ and $BOB' = n$ the angles of inclination of OA and OB respectively to the horizon. Required the angle $A'OB' = x$ made by the projections on the horizon.

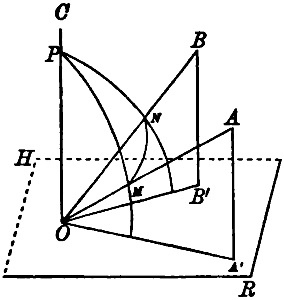


Fig. 44.

The planes of the angles of inclination AOA' and BOB' produced intersect in the line OC, which is perpendicular to the horizontal plane (Geom. § 475).

From O as a centre describe a sphere, and let its surface cut the edges of the trihedral angle O-ABC in the points M, N, and P. In the spherical triangle MNP the three sides $MN = h$, $MP = 90° - m$, $NP = 90° - n$, are known, and the spherical angle P is equal to the required angle x.

From § 52 we obtain

$$\cos \tfrac{1}{2} x = \sqrt{\cos s \cos (s - h) \sec m \sec n},$$

where $\tfrac{1}{2}(m + n + h) = s$.

§ 62. PROBLEM.

To find the distance between two places on the earth's surface (regarded as spherical), given the latitudes of the places and the difference of their longitudes.

Let M and N (Fig. 45) be the places; then their distance MN is an arc of the great circle passing through the places. Let P be the pole, AB the equator. The arcs MR and NS are the latitudes of the places, and the arc RS, or the angle MPN, is the difference of their longitudes. Let $MR = a$, $NS = b$, $RS = l$; then in the spherical triangle MNP two sides, $MP = 90 - a$, $NP = 90 - b$, and the included angle $MPN = l$, are given, and we have (from § 54)

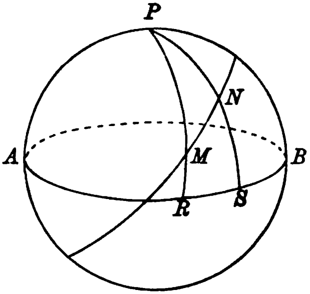


Fig. 45.

$$\tan m \quad = \cot a \cos l,$$
$$\cos MN = \sin a \sec m \sin(b + m).$$

From these equations first find m, then the arc MN, and then reduce MN to geographical miles, of which there are 60 in each degree.

§ 63. THE CELESTIAL SPHERE.

The **Celestial Sphere** is an imaginary sphere of indefinite radius, upon the concave surface of which all the heavenly bodies appear to be situated.

The **Celestial Equator,** or **Equinoctial,** is the great circle in which the plane of the earth's equator produced intersects the surface of the celestial sphere.

The **Poles** of the equinoctial are the points where the earth's axis produced cuts the surface of the celestial sphere.

The **Celestial Meridian** of an observer is the great circle in which the plane of his terrestrial meridian produced meets the surface of the celestial sphere.

Hour Circles, or **Circles of Declination,** are great circles passing through the poles, and perpendicular to the equinoctial.

The **Horizon** of an observer is the great circle in which a plane tangent to the earth's surface, at the place where he is, meets the surface of the celestial sphere.

The **Zenith** of an observer is that pole of his horizon which is exactly above his head.

Vertical Circles are great circles passing through the zenith of an observer, and perpendicular to his horizon.

The vertical circle passing through the east and west points of the horizon is called the **Prime Vertical**; that passing through the north and south points coincides with the celestial meridian.

The **Ecliptic** is a great circle of the celestial sphere, apparently traversed by the sun in one year from west to east, in consequence of the motion of the earth around the sun.

The **Equinoxes** are the points where the ecliptic cuts the equinoctial. They are distinguished as the *Vernal* equinox and the *Autumnal* equinox; the sun in his annual journey passes through the former on March 21, and through the latter on September 21.

Circles of Latitude are great circles passing through the poles of the ecliptic, and perpendicular to the plane of the ecliptic.

The angle which the ecliptic makes with the equinoctial is called the **obliquity** of the ecliptic; it is equal to 23° 27′, nearly, and is often denoted by the letter e.

These definitions are illustrated in Figs. 46 and 47. In Fig. 46, $AVBU$ is the equinoctial, P and P' its poles, $NPZS$ the celestial meridian of an observer, $NESW$ his horizon, Z his zenith, M a star, PMP' the hour circle passing through the star, $ZMDZ'$ the vertical through the star.

In Fig. 47, *AVBU* represents the equinoctial, *EVFU* the ecliptic, *P* and *Q* their respective poles, *V* the vernal equinox, *U* the autumnal equinox, *M* a star, *PMR* the hour circle through the star, *QMT* the circle of latitude through the star, and $\angle TVR = e$.

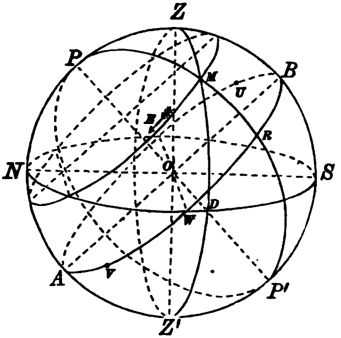


Fig. 46.

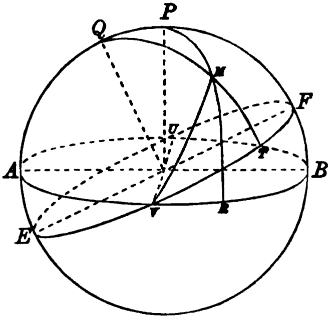


Fig. 47.

The earth's diurnal motion causes all the heavenly bodies to appear to rotate from east to west at the uniform rate of 15° per hour. If in Fig. 46 we conceive the observer placed at the centre *O*, and his zenith, horizon, and celestial meridian fixed in position, and all the heavenly bodies rotating around *PP′* as an axis from east to west at the rate of 15° per hour, we form a correct idea of the apparent diurnal motions of these bodies. When the sun or a star in its diurnal motion crosses the meridian, it is said to make a *transit* across the meridian; when it passes across the part *NWS* of the horizon, it is said to *set;* and when it passes across the part *NES*, it is said to *rise* (the effect of refraction being here neglected). Each star, as *M*, describes daily a small circle of the sphere parallel to the equinoctial, and called the **Diurnal Circle** of the star. The diurnal circle is the smaller the nearer the star is to the pole; and if there were stars at the poles *P* and *P′*, they would have no diurnal motion. To an observer north of the equator, the north pole P is *elevated* above the horizon (as shown in Fig. 46); to an observer south of the equator, the south pole P' is the elevated pole.

§ 64. SPHERICAL CO-ORDINATES.

Several systems of fixing the position of a star on the surface of the celestial sphere at any instant are in use. In each system a great circle and its pole are taken as standards of reference, and the position of the star is determined by means of two quantities called its *spherical co-ordinates*.

I. *If the horizon and the zenith* are chosen, the co-ordinates of the star are called its altitude and its azimuth.

The **Altitude** of a star is its angular distance, measured on a vertical circle, above the horizon. The complement of the altitude is called the **Zenith Distance.**

The **Azimuth** of a star is the angle at the zenith formed by the meridian of the observer and the vertical circle passing through the star, and is measured therefore by an arc of the horizon. It is usually reckoned from the north point of the horizon in north latitudes, and from the south point in south latitudes; and east or west according as the star is east or west of the meridian.

II. *If the equinoctial and its pole* are chosen, then the position of the star may be fixed by means of its declination and its hour angle.

The **Declination** of a star is its angular distance from the equinoctial, measured on an hour circle. The angular distance of the star, measured on the hour circle, *from the elevated pole* is called its **Polar Distance.**

The declination of a star, like the latitude of a place on the earth's surface, may be either north or south; but, in practical problems, while latitude is always to be considered positive, declination, if of a different name from the latitude, must be regarded as *negative*.

If the declination is negative, the polar distance is equal numerically to 90° + the declination.

The **Hour Angle** of a star is the angle at the pole formed by the meridian of the observer and the hour circle passing through the star. On account of the diurnal rotation, it is constantly changing at the rate of 15° per hour. Hour angles are reckoned from the celestial meridian, positive towards the west, and negative towards the east.

III. The *equinoctial and its pole* being still retained, we may employ as the co-ordinates of the star its declination and its right ascension.

The **Right Ascension** of a star is the arc of the equinoctial included between the vernal equinox and the point where the hour circle of the star cuts the equinoctial Right ascension is reckoned from the vernal equinox eastward from 0° to 360°.

IV. *The ecliptic and its pole* may be taken as the standards of reference. The co-ordinates of the star are then called its latitude and its longitude.

The **Latitude** of a star is its angular distance from the ecliptic measured on a circle of latitude.

The **Longitude** of a star is the arc of the ecliptic included between the vernal equinox and the point where the circle of latitude through the star cuts the ecliptic.

In problems it is useful to employ certain letters to denote these various co-ordinates. For a star M (Fig. 46), let

$l =$ latitude of the observer,
$h = DM$ = the altitude of the star,
$z = ZM$ = the zenith distance of the star,
$a = \angle PZM =$ the azimuth of the star,
$t = \angle ZPM =$ the hour angle of the star,
$d = RM$ = the declination of the star,
$p = PM$ = the polar distance of the star,
$r = VR$ = the right ascension of the star,
$u = MT$ (Fig. 47) = the latitude of the star,
$v = VT$ (Fig. 47) = the longitude of the star.

In many problems, a simple way of representing the magnitudes involved, is to project the sphere on the plane of the horizon, as shown in Fig. 48.

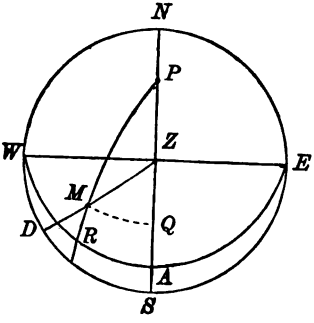


Fig. 48.

NESW is the horizon, *Z* the zenith, *NZS* the meridian, *WZE* the prime vertical, *WAE* the equinoctial projected on the plane of the horizon, *P* the elevated pole, *M* a star, *DM* its altitude, *ZM* its zenith distance, ∠*PZM* its azimuth, *MR* its declination, *PM* its polar distance, ∠*ZPM* its hour angle.

§ 65. The Astronomical Triangle.

The triangle *ZPM* (Figs. 46 and 48) is often called the *astronomical triangle*, on account of its importance in problems in Nautical Astronomy.

The side *PZ* is equal to the complement of the latitude of the observer. For (Fig. 46) the angle *ZOB* between the zenith of the observer and the celestial equator is obviously equal to his latitude, and the angle *POZ* is the complement of *ZOB*. The arc *NP* being the complement of *PZ*, it follows that *the altitude of the elevated pole is equal to the latitude of the place of observation.*

The triangle *ZPM* then (however much it may vary in shape for different positions of the star *M*), always contains the following five magnitudes:

$PZ =$ co-latitude of observer $= 90° - l$,
$ZM =$ zenith distance of star $= z$,
$PZM =$ azimuth of star $= a$,
$PM =$ polar distance of star $= p$,
$ZPM =$ hour angle of star $= t$.

A very simple relation exists between the hour angle of the sun and the local (apparent) time of day. Since the hourly rate at which the sun appears to move from east to west is 15°, and it is apparent noon when the sun is on the meridian of a place, it is evident that if hour angle = 0°, 15°, −15°, etc., time of day is noon, 1 o'clock P.M., 11 o'clock A.M., etc.

In general, if t denote the absolute value of the hour angle,

$$\text{time of day} = \frac{t}{15}\text{ P.M., or } 12 - \frac{t}{15}\text{ A.M.},$$

according as the sun is west or east of the meridian.

§ 66. PROBLEM.

Given the latitude of the observer and the altitude and azimuth of a star, to find its declination and its hour angle.

In the triangle ZPM (Fig. 48),

given $PZ = 90° - l$ = co-latitude,
$ZM = 90° - h$ = co-altitude,
$\angle PZM = a$ = azimuth;
to find $PM = 90° - d$ = polar distance,
$\angle ZPM = t$ = hour angle.

Draw $MQ \perp NS$, and put $ZQ = m$,
then, if $a < 90°$, $PQ = 90° - (l + m)$,
and if $a > 90°$, $PQ = 90° - (l - m)$;

and, by Napier's Rules,

$$\cos a = \pm \tan m \tan h,$$
$$\sin d = \cos PQ \cos MQ,$$
$$\sin h = \cos m \cos MQ;$$

whence,

$$\tan m = \pm \cot h \cos a,$$
$$\sin d = \sin h \sin (l \pm m) \sec m,$$

in which the − sign is to be used if $a > 90°$. The hour angle may then be found by means of [43], whence we have

$$\sin t = \sin a \cos h \sec d.$$

§ 67. PROBLEM.

To find the hour angle of a heavenly body when its declination, its altitude, and the latitude of the place are known.

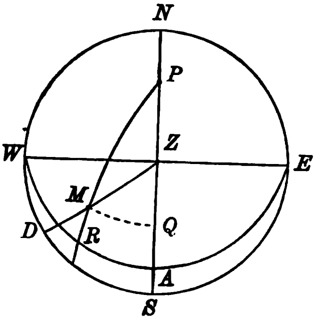


Fig. 49.

In the triangle ZPM (Fig. 49),

given $PZ = 90° - l,$
$PM = 90° - d = p,$
$ZM = 90° - h;$

required

$\angle ZPM = t.$

If, in the first formula of [46],

$$\sin \tfrac{1}{2} A = \sqrt{\sin(s-b)\,\sin(s-c)\,\csc b\,\csc c},$$

we put

$$A = t, \quad a = 90° - h, \quad b = p, \quad c = 90° - l,$$

we have

$$s - b = 90° - \tfrac{1}{2}(l + p + h), \quad s - c = \tfrac{1}{2}(l + p - h),$$

and the formula becomes

$$\sin \tfrac{1}{2} t = \pm \left[\cos \tfrac{1}{2}(l + p + h)\,\sin \tfrac{1}{2}(l + p - h)\,\sec l\,\csc p\right]^{\frac{1}{2}},$$

in which the − sign is to be taken when the body is east of the meridian.

If the body is the sun, how can the local time be found when the hour angle has been computed? (See § 65.)

§ 68. PROBLEM.

To find the altitude and azimuth of a celestial body, when its declination, its hour angle, and the latitude of the place are known.

In the triangle ZPM (Fig. 49),

given $PZ = 90^\circ - l,$

$PM = 90^\circ - d = p,$

$\angle ZPM = t;$

required $ZM = 90^\circ - h,$

$\angle PZM = a.$

Here there are given two sides and the included angle. Placing $PQ = m$, and proceeding as in § 66, we obtain

$$\begin{aligned} \tan m &= \cot d \cos t, \\ \sin h &= \sin(l+m) \sin d \sec m, \\ \tan a &= \sec(l+m) \tan t \sin m, \end{aligned}$$

in the last of which formulas a must be marked E. or W., to agree with the hour angle.

§ 69. PROBLEM.

To find the latitude of the place when the altitude of a celestial body, its declination, and its hour angle are known.

In the triangle ZPM (Fig. 49),

given $ZM = 90^\circ - h,$

$PM = 90^\circ - d,$

$\angle ZPM = t;$

required $PZ = 90^\circ - l.$

Let $PQ = m, \quad ZQ = n.$

Then, by Napier's Rules,

$$\begin{aligned}\cos t &= \tan m \tan d,\\ \sin h &= \cos n \cos MQ,\\ \sin d &= \cos m \cos MQ;\end{aligned}$$

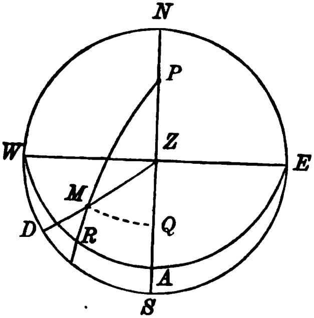


Fig. 49.

whence,

$$\begin{aligned}\tan m &= \cot d \cos t,\\ \cos n &= \cos m \sin h \csc d,\end{aligned}$$

and it is evident from the figure that

$$l = 90° - (m \pm n),$$

in which the sign + or the sign − is to be taken according as the body and the elevated pole are *on the same side* of the prime vertical or *on opposite sides*.

In fact, both values of l may be possible for the same altitude and hour angle; but, unless n is very small, the two values will differ largely from each other, so that the observer has no difficulty in deciding which of them should be taken.

§ 70. Problem.

Given the declination, the right ascension of a star, and the obliquity of the ecliptic, to find the latitude and the longitude of the star.

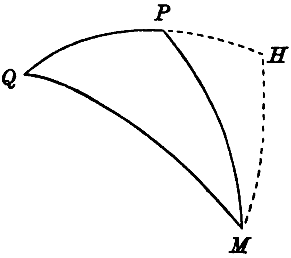


Fig. 50.

Let M (Fig. 50) be the star, P be the pole of the equinoctial, and Q the pole of the ecliptic.

Then, on the triangle PMQ,

given $PQ = e = 23° 27'$,

$PM = 90° - d$,

$\angle MPQ = 90° + r$ (see Fig. 47);

required $QM = 90° - u$, and $\angle PQM = 90° - v$ (see Fig. 47).

In this case, also, two sides and the included angle are given. Draw $MH \perp PQ$, and meeting it produced at H, and let $PH = n$.

By Napier's Rules,

$$\begin{aligned} \sin r &= \tan n \tan d, \\ \sin u &= \cos (e + n) \cos MH, \\ \sin d &= \cos n \cos MH, \\ \sin (e + n) &= \tan v \tan MH, \\ \sin n &= \tan r \tan MH; \end{aligned}$$

whence,

$$\begin{aligned} \tan n &= \cot d \sin r, \\ \sin u &= \sin d \cos (e + n) \sec n, \\ \tan v &= \tan r \sin (e + n) \csc n. \end{aligned}$$

To avoid obtaining u from its sine we may proceed as follows:

From the last two equations we have, by division,

$$\sin u = \tan v \cot (e + n) \sin d \cot r \tan n.$$

By taking MH as middle part, successively, in the triangles MQH and MPH, we obtain

$$\cos u \cos v = \cos d \cos r;$$

whence, $\cos u = \sec v \cos d \cos r.$

From these values of $\sin u$ and $\cos u$ we obtain, by division,

$$\tan u = \sin v \cot (e + n) \tan d \csc r \tan n.$$

From the relation

$$\sin r = \tan n \tan d,$$

it follows that $\tan d \csc r \tan n = 1$.

Therefore $\tan u = \sin v \cot (e + n)$,

a formula by which u can be easily found after v has been computed.

EXERCISE XXXII.

1. Find the dihedral angle made by the lateral faces of a regular ten-sided pyramid; given the angle $A = 18°$, made at the vertex by two adjacent lateral edges.

2. Through the foot of a rod which makes the angle A with a plane, a straight line is drawn in the plane. This line makes the angle B with the projection of the rod upon the plane. What angle does this line make with the rod?

3. Find the volume V of an oblique parallelopipedon; given the three unequal edges a, b, c, and the three angles l, m, n, which the edges make with one another.

4. The continent of Asia has nearly the shape of an equilateral triangle, the vertices being the East Cape, Cape Romania, and the Promontory of Baba. Assuming each side of this triangle to be 4800 geographical miles, and the earth's radius to be 3440 geographical miles, find the area of the triangle: (i.) regarded as a plane triangle; (ii.) regarded as a spherical triangle.

5. A ship sails from a harbor in latitude l, and keeps on the arc of a great circle. Her *course* (or angle between the direction in which she sails and the meridian) at starting is a. Find where she will cross the equator, her course at the equator, and the distance she has sailed.

6. Two places have the same latitude l, and their distance apart, measured on an arc of a great circle, is d. How much greater is the arc of the parallel of latitude between the places than the arc of the great circle? Compute the results for $l = 45°$, $d = 90°$.

7. The shortest distance d between two places and their latitudes l and l' are known. Find the difference between their longitudes.

8. Given the latitudes and longitudes of three places on the earth's surface, and also the radius of the earth; show how to find the area of the spherical triangle formed by arcs of great circles passing through the places.

9. The distance between Paris and Berlin (that is, the arc of a great circle between these places) is equal to 472 geographical miles. The latitude of Paris is 48° 50′ 13″; that of Berlin, 52° 30′ 16″. When it is noon at Paris what time is it at Berlin?

NOTE. Owing to the apparent motion of the sun, the local time over the earth's surface at any instant varies at the rate of one hour for 15° of longitude; and the more *easterly* the place, the *later* the local time.

10. The altitude of the pole being 45°, I see a star on the horizon and observe its azimuth to be 45°; find its polar distance.

11. Given the latitude l of the observer, and the declination d of the sun; find the local time (apparent solar time) of sunrise and sunset, and also the azimuth of the sun at these times (refraction being neglected). When and where does the sun rise on the longest day of the year (at which time $d = +23° 27'$) in Boston ($l = 42° 21'$), and what is the length of the day from sunrise to sunset? Also, find when and where the sun rises in Boston on the shortest day of the year (when $d = -23° 27'$), and the length of this day.

12. When is the solution of the problem in Example 11 impossible, and for what places is the solution impossible?

13. Given the latitude of a place and the sun's declination; find his altitude and azimuth at 6 o'clock A.M. (neglecting refraction). Compute the results for the longest day of the year at Munich ($l = 48° 9'$).

14. How does the altitude of the sun at 6 A.M. *on a given day* change as we go from the equator to the pole? At what time of the year is it a maximum *at a given place?* (Given $\sin h = \sin l \sin d$.)

15. Given the latitude of a place north of the equator, and the declination of the sun; find the time of day when the sun bears due east and due west. Compute the results for the longest day at St. Petersburg ($l = 59° 56'$).

16. Apply the general result in Example 15 ($\cos t = \cot l \tan d$) to the case when the days and nights are equal in length (that is, when $d = 0°$). Why can the sun in summer never be due east before 6 A.M., or due west after 6 P.M.? How does the time of bearing due east and due west change with the declination of the sun? Apply the general result to the cases where $l < d$ and $l = d$. What does it become at the north pole?

17. Given the sun's declination and his altitude when he bears due east; find the latitude of the observer.

18. At a point O in a horizontal plane MN a staff OA is fixed, so that its angle of inclination AOB with the plane is equal to the latitude of the place, 51° 30′ N., and the direction OB is due north. What angle will OB make with the shadow of OA on the plane, at 1 P.M., when the sun is on the equinoctial?

19. What is the direction of a wall in latitude 52° 30′ N. which casts no shadow at 6 A.M. on the longest day of the year?

20. At a certain place the sun is observed to rise exactly in the north-east point on the longest day of the year; find the latitude of the place.

21. Find the latitude of the place at which the sun sets at 10 o'clock on the longest day.

22. To what does the general formula for the hour angle, in § 67, reduce when (i.) $h = 0°$, (ii.) $l = 0°$ and $d = 0°$, (iii.) l or $d = 90°$?

23. What does the general formula for the azimuth of a celestial body, in § 68, become when $t = 90° = 6$ hours?

24. Show that the formulas of § 69, if $t = 90°$, lead to the equation $\sin l = \sin h \csc d$; and that if $d = 0°$, they lead to the equation $\cos l = \sin h \sec t$.

25. Given latitude of place 52° 30′ 16″, declination of star 38°, its hour angle 28° 17′ 15″; find its altitude.

26. Given latitude of place 51° 19′ 20″, polar distance of star 67° 59′ 5′, its hour angle 15° 8′ 12″; find its altitude and its azimuth.

27. Given the declination of a star 7° 54′, its altitude 22° 45′ 12″, its azimuth 129° 45′ 37″; find its hour angle and the latitude of the observer.

28. Given the longitude v of the sun, and the obliquity of the ecliptic $e = 23° 27′$; find the declination d, and the right ascension r.

29. Given the obliquity of the ecliptic $e = 23° 27′$, the latitude of a star 51°, its longitude 315°; find its declination and its right ascension.

30. Given the latitude of place 44° 50′ 14″, the azimuth of a star 138° 58′ 43″, and its hour angle 20°; find its declination.

31. Given latitude of place 51° 31′ 48″, altitude of sun west of the meridian 35° 14′ 27″, its declination +21° 27′; find the local apparent time.

32. Given latitude of place l, the polar distance p of a star, and its altitude h; find its azimuth a.

APPENDIX.

FORMULAS.

$$\left.\begin{array}{ll} \textbf{1.} & \sin^2 A+\cos^2 A=1. \\[2ex] \textbf{2.} & \tan A=\dfrac{\sin A}{\cos A}. \\[2ex] \textbf{3.} & \left\{\begin{array}{l}\sin A\times\csc A=1.\\ \cos A\times\sec A=1.\\ \tan A\times\cot A=1.\end{array}\right. \end{array}\right\}\ \S\ 5.$$

$$\left.\begin{array}{ll} \textbf{4.} & \sin(x+y)=\sin x\cos y+\cos x\sin y. \\[2ex] \textbf{5.} & \cos(x+y)=\cos x\cos y-\sin x\sin y. \\[2ex] \textbf{6.} & \tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}. \\[2ex] \textbf{7.} & \cot(x+y)=\dfrac{\cot x\cot y-1}{\cot x+\cot y}. \end{array}\right\}\ \S\ 31.$$

$$\left.\begin{array}{ll} \textbf{8.} & \sin(x-y)=\sin x\cos y-\cos x\sin y. \\[2ex] \textbf{9.} & \cos(x-y)=\cos x\cos y+\sin x\sin y. \\[2ex] \textbf{10.} & \tan(x-y)=\dfrac{\tan x-\tan y}{1+\tan x\tan y}. \\[2ex] \textbf{11.} & \cot(x-y)=\dfrac{\cot x\cot y+1}{\cot y-\cot x}. \end{array}\right\}\ \S\ 32.$$

$$\left.\begin{array}{ll} \textbf{12.} & \sin 2x=2\sin x\cos x. \\[2ex] \textbf{13.} & \cos 2x=\cos^2 x-\sin^2 x. \end{array}\right\}\ \S\ 33.$$

$$\left.\begin{aligned}
&\textbf{14.}\quad \tan 2x = \frac{2\tan x}{1-\tan^2 x}.\\[2ex]
&\textbf{15.}\quad \cot 2x = \frac{\cot^2 x - 1}{2\cot x}.
\end{aligned}\right\}\ \S\ 33.$$

$$\left.\begin{aligned}
&\textbf{16.}\quad \sin \tfrac{1}{2} z = \pm\sqrt{\frac{1-\cos z}{2}}.\\[2ex]
&\textbf{17.}\quad \cos \tfrac{1}{2} z = \pm\sqrt{\frac{1+\cos z}{2}}.\\[2ex]
&\textbf{18.}\quad \tan \tfrac{1}{2} z = \pm\sqrt{\frac{1-\cos z}{1+\cos z}}.\\[2ex]
&\textbf{19.}\quad \cot \tfrac{1}{2} z = \pm\sqrt{\frac{1+\cos z}{1-\cos z}}.
\end{aligned}\right\}\ \S\ 34.$$

$$\left.\begin{aligned}
&\textbf{20.}\quad \sin A + \sin B = 2 \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B).\\[2ex]
&\textbf{21.}\quad \sin A - \sin B = 2 \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B).\\[2ex]
&\textbf{22.}\quad \cos A + \cos B = 2 \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B).\\[2ex]
&\textbf{23.}\quad \cos A - \cos B = -2 \sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B).\\[2ex]
&\textbf{24.}\quad \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.
\end{aligned}\right\}\ \S\ 35.$$

25. $\dfrac{a}{b} = \dfrac{\sin A}{\sin B}.$ § 36.

26. $a^2 = b^2 + c^2 - 2bc \cos A.$ § 37.

27. $\dfrac{a-b}{a+b} = \dfrac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}.$ § 38.

28. $\sin \tfrac{1}{2} A = \sqrt{\dfrac{(s-b)(s-c)}{bc}}.$ § 43.

$$\left.\begin{array}{ll}
\textbf{29.} & \cos\tfrac{1}{2}A=\sqrt{\dfrac{s(s-a)}{bc}}.\\[2ex]
\textbf{30.} & \tan\tfrac{1}{2}A=\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}.\\[2ex]
\textbf{31.} & \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}=r.\\[2ex]
\textbf{32.} & \tan\tfrac{1}{2}A=\dfrac{r}{s-a}.
\end{array}\right\}\ \S\ 43.$$

$$\left.\begin{array}{ll}
\textbf{33.} & F=\tfrac{1}{2}ac\sin B.\\[2ex]
\textbf{34.} & F=\sqrt{s(s-a)(s-b)(s-c)}.\\[2ex]
\textbf{35.} & F=\dfrac{abc}{4R}.\\[2ex]
\textbf{36.} & F=\tfrac{1}{2}r(a+b+c)=rs.
\end{array}\right\}\ \S\ 44.$$

SPHERICAL TRIGONOMETRY.

$$\left.\begin{array}{ll}
\textbf{37.} & \cos c=\cos a\cos b.\\[2ex]
\textbf{38.} & \left\{\begin{array}{l}\sin a=\sin c\sin A.\\ \sin b=\sin c\sin B.\end{array}\right.\\[2ex]
\textbf{39.} & \left\{\begin{array}{l}\cos A=\tan b\cot c.\\ \cos B=\tan a\cot c.\end{array}\right.\\[2ex]
\textbf{40.} & \left\{\begin{array}{l}\cos A=\cos a\sin B.\\ \cos B=\cos b\sin A.\end{array}\right.\\[2ex]
\textbf{41.} & \left\{\begin{array}{l}\sin b=\tan a\cot A.\\ \sin a=\tan b\cot B.\end{array}\right.\\[2ex]
\textbf{42.} & \cos c=\cot A\cot B.
\end{array}\right\}\ \S\ 46.$$

$$\textbf{43.}\ \left\{\begin{array}{l}\sin a\sin B=\sin b\sin A.\\ \sin a\sin C=\sin c\sin A.\\ \sin b\sin C=\sin c\sin B.\end{array}\right\}\ \S\ 51.$$

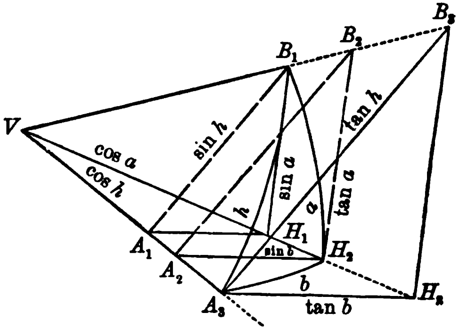
$$\left.\begin{array}{l}
\textbf{44.}\ \begin{cases}\cos a = \cos b \cos c + \sin b \sin c \cos A. \\ \cos b = \cos a \cos c + \sin a \sin c \cos B. \\ \cos c = \cos a \cos b + \sin a \sin b \cos C.\end{cases} \\[2ex]
\textbf{45.}\ \begin{cases}\cos A = -\cos B \cos C + \sin B \sin C \cos a. \\ \cos B = -\cos A \cos C + \sin A \sin C \cos b. \\ \cos C = -\cos A \cos B + \sin A \sin B \cos c.\end{cases}
\end{array}\right\}\ \S\ 51.$$

$$\left.\begin{array}{l}
\textbf{46.}\ \begin{cases}\sin \tfrac{1}{2} A = \sqrt{\sin (s-b) \sin (s-c) \csc b \csc c}. \\ \cos \tfrac{1}{2} A = \sqrt{\sin s \sin (s-a) \csc b \csc c}. \\ \tan \tfrac{1}{2} A = \sqrt{\csc s \csc (s-a) \sin (s-b) \sin (s-c)}.\end{cases} \\[2ex]
\textbf{47.}\ \begin{cases}\sin \tfrac{1}{2} a = \sqrt{-\cos S \cos (S-A) \csc B \csc C}. \\ \cos \tfrac{1}{2} a = \sqrt{\cos (S-B) \cos (S-C) \csc B \csc C}. \\ \tan \tfrac{1}{2} a = \sqrt{-\cos S \cos (S-A) \sec (S-B) \sec (S-C)}.\end{cases}
\end{array}\right\}\ \S\ 52.$$

$$\left.\begin{array}{l}
\textbf{48.}\ \begin{cases}\cos \tfrac{1}{2} (A+B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2} (a+b) \sin \tfrac{1}{2} C. \\ \sin \tfrac{1}{2} (A+B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2} (a-b) \cos \tfrac{1}{2} C. \\ \cos \tfrac{1}{2} (A-B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2} (a+b) \sin \tfrac{1}{2} C. \\ \sin \tfrac{1}{2} (A-B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2} (a-b) \cos \tfrac{1}{2} C.\end{cases} \\[2ex]
\textbf{49.}\ \begin{cases}\tan \tfrac{1}{2} (A+B) = \dfrac{\cos \frac{1}{2} (a-b)}{\cos \frac{1}{2} (a+b)} \cot \tfrac{1}{2} C. \\[2ex] \tan \tfrac{1}{2} (A-B) = \dfrac{\sin \frac{1}{2} (a-b)}{\sin \frac{1}{2} (a+b)} \cot \tfrac{1}{2} C. \\[2ex] \tan \tfrac{1}{2} (a+b) = \dfrac{\cos \frac{1}{2} (A-B)}{\cos \frac{1}{2} (A+B)} \tan \tfrac{1}{2} c. \\[2ex] \tan \tfrac{1}{2} (a-b) = \dfrac{\sin \frac{1}{2} (A-B)}{\sin \frac{1}{2} (A+B)} \tan \tfrac{1}{2} c.\end{cases}
\end{array}\right\}\ \S\ 53.$$

$$\left.\begin{array}{l}
\textbf{50.}\quad F = \dfrac{E}{180^\circ} \pi R^2. \\[3ex]
\textbf{51.}\quad \tan^2 \tfrac{1}{4} E = \tan \tfrac{1}{2} s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c).
\end{array}\right\}\ \S\ 60.$$

Prof. Blakslee's construction by which the direction ratios for plane right triangles give directly from a figure the analogies for a right trihedral or for a right spherical triangle.



The construction consists of two parts.

(*a*) Lay off from the vertex V a unit's distance on each edge.

(*b*) Pass through the three extremities of these distances three planes perpendicular to one of the edges, as VA. Now these three parallel planes will cut out three similar right triangles. The first being constructed in either of the two usual ways, the construction of the others is evident.

Since the plane angles A_1, A_2, A_3 all equal the dihedral A, and the nine right triangles in the three faces give the values in the figure, we have:

(1) $\sin A = \sin a : \sin h$; similarly, $\sin B = \sin b : \sin h$.

(2) $\cos A = \tan b : \tan h$; similarly, $\cos B = \tan a : \tan h$.

(3) $\tan A = \tan a : \sin b$; similarly, $\tan B = \tan b : \sin a$.

(4) $\cos h = \cos a \ \cos b$; (by 3) $= \cot A \cot B$.

(5) $\sin A = \cos B : \cos b$; $\sin B = \cos A : \cos a$.

Note. If a sphere of unit radius be described about V as a centre, the three faces will cut out a right spherical triangle, having the sides a, b, and h, and angles A, B, and H. The above formulas are thus seen to be the analogies of:

(1) $\sin A = a : h$; $\sin B = b : h$.

(2) $\cos A = b : h$; $\cos B = a : h$.

(3) $\tan A = a : b$; $\tan B = b : a$.

(4) $h^2 = a^2 + b^2$; $1 = \sin^2 + \cos^2$; $1 = \cot A \cot B$.

(5) $\sin A = \cos B$; $\sin B = \cos A$.

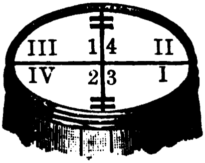
Napier's rules give only the following, which follow from the analogies as numbered:

$$\begin{array}{l} \text{By} \\ (1) \end{array} \left\{ \begin{array}{l} \sin a = \sin A \sin h = \tan b \cot B \\ \sin b = \sin B \sin h = \tan a \cot A \end{array} \right\} (3)$$

$$(5) \left\{ \begin{array}{l} \cos A = \sin B \cos a = \tan b \cot h \\ \cos B = \sin A \cos b = \tan a \cot h \end{array} \right\} (2)$$

$$(4) \left\{ \cos h = \cos a \cos b = \cot A \cot B \right\} (4)$$

THE GAUSS EQUATIONS.



$$\cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a + b) \sin \tfrac{1}{2} C.$$

$$\sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a - b) \cos \tfrac{1}{2} C.$$

$$\cos \tfrac{1}{2}(A - B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a + b) \sin \tfrac{1}{2} C.$$

$$\sin \tfrac{1}{2}(A - B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a - b) \cos \tfrac{1}{2} C.$$

$$f \tfrac{1}{2}(A \pm B) \quad f \tfrac{1}{2} c = \quad f \tfrac{1}{2}(a \pm b) \quad f \tfrac{1}{2} C.$$

RULE I. sin in (I.) gives − in (3), and conversely.
cos in (I.) gives + in (3), and conversely.

RULE II. Functions have same names in (2) and (3).
Functions have co-names in (4) and (1).

SURVEYING AND NAVIGATION.

BY

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BOSTON, U.S.A.:
GINN & COMPANY, PUBLISHERS.
1892.

Entered, according to Act of Congress, in the year 1882, by
G. A. WENTWORTH,
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TYPOGRAPHY BY J. S. CUSHING & CO., BOSTON, U.S.A.

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PREFACE.

THE object of this brief work on Surveying and Navigation is to present these subjects in a clear and intelligible way, according to the best methods in actual use; and also to present them in so small a compass that students in general may find the time to acquire a competent knowledge of these very interesting and important studies.

The author is under special obligation to Professor James L. Patterson of Lawrenceville, N.J., who has done a large share of the work on both the Surveying and the Navigation.

G. A. WENTWORTH.

PHILLIPS EXETER ACADEMY,
September, 1883.

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SURVEYING.

CHAPTER I.

DEFINITIONS. INSTRUMENTS AND THEIR USES.

§ 1. DEFINITIONS.

Surveying is the art of determining and representing distances, areas, and the relative position of points upon the surface of the earth.

In plane surveying, the portion surveyed is considered as a plane.

In geodetic surveying, the curvature of the earth is regarded.

A **Plumb-Line** is a cord with a weight attached and freely suspended.

A **Vertical Line** is a line having the direction of the plumb-line.

A **Vertical Plane** is a plane embracing a vertical line.

A **Horizontal Plane** is a plane perpendicular to a vertical line.

A **Horizontal Line** is a line in a horizontal plane.

A **Horizontal Angle** is an angle the sides of which are in a horizontal plane.

A **Vertical Angle** is an angle the sides of which are in a vertical plane. If one side of a vertical angle is horizontal, and the other ascends, it is an *angle of elevation;* if one side is horizontal, and the other descends, it is an *angle of depression.*

The **Magnetic Meridian** is the direction which a bar-magnet assumes when freely supported in a horizontal position.

a plumb-line, or a slender staff, or, less accurately, by dropping a pin (heavy end downwards), the point vertically under the raised end of the chain may be determined. If the slope is considerable, half a chain or less may be used.

To construct a perpendicular with a chain:

1. When the point through which the perpendicular is to pass is in the line:

Let *AB* (Fig. 1) represent the line, and *P* the point. Measure from *P* to the right or left, *PC* = 40 links, and place a stake at *C*. Let one end of the chain be held at *P*, and the end of the eightieth link at *C*; then, taking the chain at the end of the thirtieth link from *P*, draw it so that the portions *DC* and *DP* are tightly stretched, and place a stake at *D*. *DP* will be the perpendicular required. (Why?)

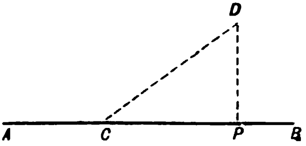


Fig. 1.

2. When the point is without the line:

Let *AB* (Fig. 1) be the line, and *D* the point. Take *C* any point in the line, and stretch the chain between *D* and *C*; then, let the middle of the part of the chain between *C* and *D* be held in place, and swing the end at *D* around until it meets the line in *P*. *DP* will be the perpendicular required. (Why?)

§ 4. Obstacles to Chaining.

1. When a tree, building, or other obstacle is encountered in measuring or extending a line, it may be passed by an offset in the following manner:

Let *O* (Fig. 2) represent a building on the line *AD*. At *B* erect *BE* perpendicular to *AB*; at *E* erect *EF* perpendicular to *BE*; at *F* erect *FC* = *BE* perpendicular to *EF*: then, *CD* perpendicular to *FC* will be in the required line, and *AB* + *EF* + *CD* = *AD*. By constructing two other perpendiculars, *B′E′* and *F′C′*, the accuracy of the work will be increased.

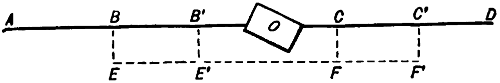


Fig. 2.

2. To measure across a body of water:

Let it be required to measure the line *ABCD* (Fig. 3) crossing a river between *B* and *C*. Measure $BE = 400$ links; at *E* erect the perpendicular $EF = 600$ links; at *B* erect the perpendicular $BG = 300$ links. Place a stake at *C*, the intersection of *AD* and *FG* beyond the river.

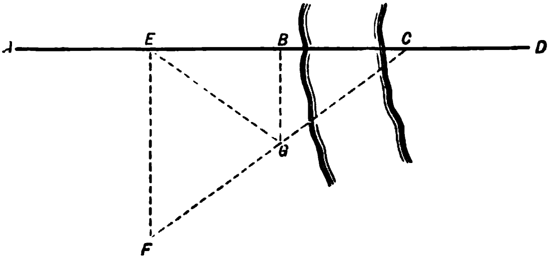


Fig. 3.

Then $BC = 400$ links. For, by similar triangles, $EF : BG :: CE : CB$. But $EF = 2\,BG$; hence, $CE = 2\,CB$, and $CB = BE = 400$ links. *EG* and *FG* should be measured, in order to test the accuracy of the work. $EG = FG = 500$ links.

Instead of the above distances, any convenient distances may be taken. For, if $EF = 2\,BG$, then $CB = BE$, and $EG = FG = \sqrt{\overline{EB}^2 + \overline{BG}^2}$.

3. To measure a line the end of which is invisible from the beginning, and intermediate points unknown:

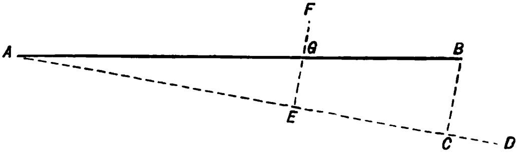


Fig. 4.

Let *AB* (Fig. 4) represent the line. Set up a flag-staff at *D*, beyond *B* and visible from *A*. From *B* let fall *BC* perpendicular to *AD*. Measure *AC* and *BC*. Then

$$AB = \sqrt{\overline{AC}^2 + \overline{BC}^2}.$$

To find intermediate points on *AB*:

At any point *E* on *AC* erect *EF* perpendicular to *AC*, and determine *EG* by the proportion $AC : CB :: AE : EG$. *G* will be a point on *AB*

The line *AD* is called a **Random Line.**

THE MEASUREMENT OF ANGLES.

§ 5. THE VERNIER COMPASS.*

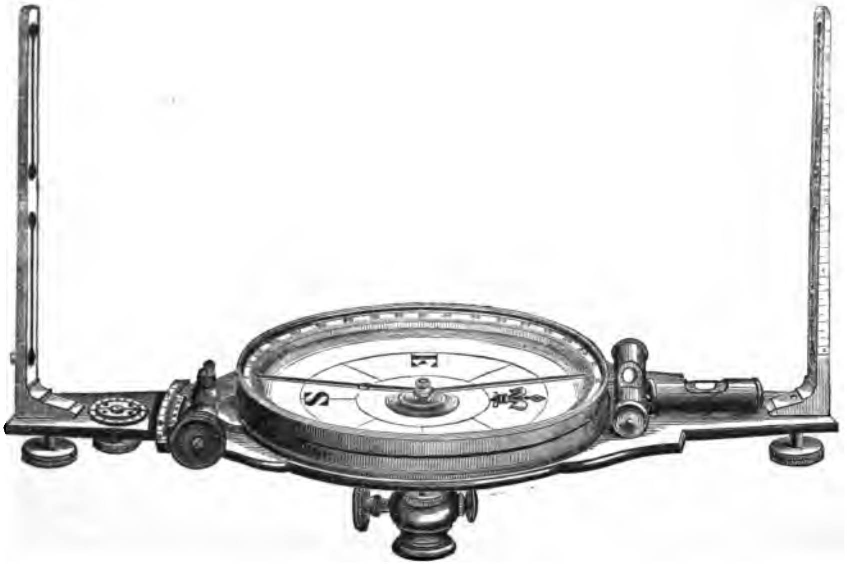
The **Vernier Compass** is shown on the following page.

The *compass circle* is divided into half-degrees, and is figured from 0° to 90° each way from the north and south points. In the centre of the compass circle is the *pivot* which supports the *magnetic needle.* The needle may be lifted from the pivot by a spring and pressed against the glass covering of the compass circle, when the instrument is not in use. The *main plate* moves around the compass circle through a small arc, read by the *vernier*, for the purpose of allowing for the variation of the needle (§ 23). The *sight standards* at the extremities of the main plate have fine slits nearly their whole length, with circular openings at intervals; on the edges of the north standard *tangent scales* are placed; and on the outside of the south standard two *eye-pieces* at the same distance from the main plate as the zeros of the tangent scales, respectively. The *telescopic sight* (a recent improvement by the Messrs. Gurley), consists of a small telescope attached to the south standard. The main plate is furnished with two *spirit levels* at right angles, and turns horizontally upon the upper end of the *ball spindle*, the lower end of which rests in a spherical socket in the top of the *tripod* or *Jacob's staff* which supports the instrument. From the centre of the plate at the top of the tripod a plumb-bob is suspended by which the centre of the compass can be placed directly over a definite point on the ground.

---

* The instruments described on this and the following pages are adjusted by the maker. If they should require readjustment, full directions will be found in the manual furnished with the instruments.

The manual published by Messrs. W. & L. E. GURLEY, Troy, N.Y., has been freely used, by permission, in describing these instruments.



THE VERNIER COMPASS.

§ 6. USES OF THE COMPASS.

To take the bearing* of a line. Place the instrument so that the bob will be directly over one end of the line, and level by pressing with the hands on the main plate until the bubbles are brought to the middle of the spirit levels. Turn the south end of the instrument toward you, and sight at the flag-staff at the other end of the line. Read the bearing from the north end of the needle. First, write N. or S. according as the north end of the needle is nearer N. or S. of the compass circle; secondly, write the number of degrees between the north end of the needle and the nearest zero mark; and thirdly, write E. or W. according as the north end of the needle is nearer E. or W. of the compass circle.

In Fig. 5 the bearing would be N. 45° W.
In Fig. 6 the bearing would be S. 45° W.
In Fig. 7 the bearing would be S. 30° E.
In Fig. 8 the bearing would be N. 60° E.

If the needle coincides with the N.S. or E.W. line, the bearing would be N., S., E., or W., according as the north end of the needle is over N., S., E., or W.

As the compass circle is divided into half-degrees, the bearing may be determined pretty accurately to quarter-degrees.

When a fence or other obstruction interferes with placing the instrument over the line, it may be placed at one side, the flag-staff being placed at an equal distance from the line.

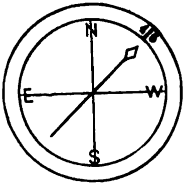


Fig. 5.

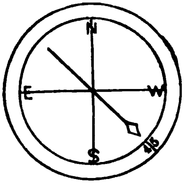


Fig. 6.

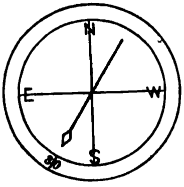


Fig. 7.

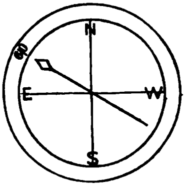


Fig. 8.

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* The magnetic bearing is meant unless otherwise specified.

Local Disturbances. Before a bearing is recorded, care should be exercised that the chain, pins, and other instruments that would affect the direction of the needle, are removed from the vicinity of the compass. Even after the greatest care in this respect is exercised, the direction of the needle is often affected by iron ore, ferruginous rocks, etc.

Reverse Bearings. When the bearing of a line has been taken, the instrument should be removed to the other end of the line and the reverse bearing taken. The number of degrees should be the same as before, but the letters should be reversed.

To take the bearing of a line one end of which cannot be seen from the other. Run a random line (§ 4, 3); then (Fig. 4),

$$\tan CAB = \frac{BC}{AC};$$

whence, the angle *CAB* may be found. This angle combined with the bearing of the random line will give the bearing required.

Another method will be given in § 19.

To measure a horizontal angle by means of the needle. Place the compass over the vertex of the angle, take the bearing of each side separately, and combine these bearings.

To measure angles of elevation. Bring the south end of the compass towards you, place the eye at the lower eye-piece, and with the hand hold a card on the front side of the north sight, so that its top edge will be at right angles to the divided edge and coincide with the zero mark; then, sighting over the top of the card, note upon a flag-staff the height cut by the line of sight; move the staff up the elevation, and carry the card along the edge of the sight until the line of sight again cuts the same height on the staff; read off the degrees of the tangent scale passed over by the card.

To measure angles of depression. Proceed in the same manner as above, using the eye-piece and tangent scale on the opposite sides of the sights, and reading from the top of the sight.

§ 7. Verniers.

First form. Let AB (Fig. 9) represent a portion of a rod for measuring heights (§ 32). The graduation to feet and hundredths of a foot begins at the lower end, which rests on the ground when the rod is in use. The line extending nearly across the rod at the bottom of the portion shown marks the beginning of the fourth foot. The face of the rod is divided into four columns: in the first is written the number of feet; in the second, the number of tenths; and in the third, the number of hundredths.

It is evident that, with the arrangement just described, heights could be measured only to hundredths of a foot. To enable us to find the height more precisely, a contrivance called a **Vernier** is used. This is shown at the right of the rod. It consists of a piece of metal or wood, the graduated part of which is $\frac{11}{100}$ of a foot in length; and this is divided into ten equal parts. Hence, one division of the vernier $= \frac{1}{10}$ of $\frac{11}{100} = \frac{11}{1000}$ of a foot; and one division of the vernier exceeds one division of the rod by $\frac{11}{1000} - \frac{1}{100} = \frac{1}{1000}$ of a foot.

The vernier slides along the face or side of the rod.

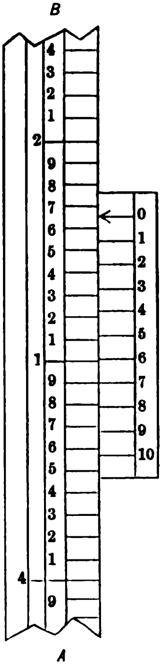


Fig. 9.

To use the vernier, place the lower end of the rod upon the ground, and move the vernier until its index or zero mark is opposite the point whose distance from the ground is desired. In the figure, the height of the index of the vernier is evidently 4.16 feet, increased by the distance of the index above the next lower line (4.16) of the rod. We shall now determine this distance.

Observe which line of the vernier is exactly opposite a line of the rod. In this case, the line of the vernier numbered 7 is opposite a line of the rod. Then, since each division of the vernier exceeds each division of the rod by $\frac{1}{1000}$ of a foot,

6 of the vernier is $\frac{1}{1000}$ of a foot above the next lower line of the rod.
5 of the vernier is $\frac{2}{1000}$ of a foot above the next lower line of the rod.
4 of the vernier is $\frac{3}{1000}$ of a foot above the next lower line of the rod.
3 of the vernier is $\frac{4}{1000}$ of a foot above the next lower line of the rod.
2 of the vernier is $\frac{5}{1000}$ of a foot above the next lower line of the rod.
1 of the vernier is $\frac{6}{1000}$ of a foot above the next lower line of the rod.
0 of the vernier is $\frac{7}{1000}$ of a foot above the next lower line of the rod.

Hence, the required reading is $4.16 + .007 = 4.167$ feet.

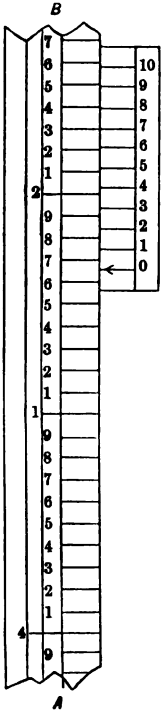


Fig. 10.

In general, the following rule is evident:

Move the vernier until its zero line is at the required height; read the height to the nearest hundredth below the index, and write in the thousandths' place the number of the division line of the vernier which stands opposite any line of the rod.

Second form. In this form (Fig. 10) the graduated part of the vernier is $\frac{9}{100}$ of a foot in length, and is divided into ten equal parts. Hence, one division of the vernier $= \frac{1}{10}$ of $\frac{9}{100} = \frac{9}{1000}$ of a foot; and one division of the vernier is less than one division of the rod by $\frac{1}{100} - \frac{9}{1000} = \frac{1}{1000}$ of a foot.

The height of the index of the vernier in Fig. 10 is 4.16 feet, increased by the distance of the index from the next lower line (4.16) of the rod. We shall now determine this distance.

We observe that the line of the vernier numbered 7 stands exactly opposite a line (3) of the rod. Hence,

6 of the vernier is $\frac{1}{1000}$ of a foot above the next lower line of the rod.
5 of the vernier is $\frac{2}{1000}$ of a foot above the next lower line of the rod.
4 of the vernier is $\frac{3}{1000}$ of a foot above the next lower line of the rod.
3 of the vernier is $\frac{4}{1000}$ of a foot above the next lower line of the rod.
2 of the vernier is $\frac{5}{1000}$ of a foot above the next lower line of the rod.
1 of the vernier is $\frac{6}{1000}$ of a foot above the next lower line of the rod.
0 of the vernier is $\frac{7}{1000}$ of a foot above the next lower line of the rod.

Hence, the required reading is $4.16 + .007 = 4.167$ feet; and the rule is evidently the same as for the first form.

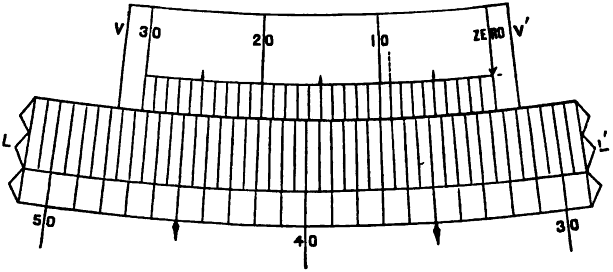


Fig. 11.

Compass Verniers. Let LL' (Fig. 11) represent the limb of the compass graduated to half-degrees, and VV' the vernier divided into thirty equal spaces, equal to twenty-nine spaces of the limb. Then one space of the vernier is less than one space of the limb by 1′, and the reading may be obtained to single minutes.

In Fig. 11 the index or zero of the vernier stands between 32° and 32° 30′, and the line of the vernier marked 9 coincides with a line of the limb. Hence, the reading is 32° 9′.

When the index moves from the zero line of the limb in a direction contrary to that in which the numbers of the limb run, the number of minutes obtained as above must be subtracted from 30′ to obtain the minutes required.

If, however, the vernier be made double, that is, if it have thirty spaces on each side of the zero line, it is always read directly. The usual form of the double vernier, shown in

Fig. 12, has only fifteen spaces on each side of the zero line. When the vernier is turned to the *right* less than 15′ past a division line of the limb, read the lower figures on the *left* of the zero line at any coincidence; if moved more than 15′ past a division line of the limb, read the upper figures on the *right* of the zero line at any coincidence; and *vice versa.*

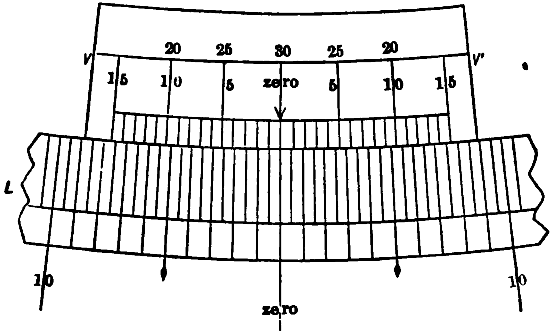


Fig. 12.

Uses of the Compass Vernier. The most important use of the vernier of the vernier compass is in setting off the variation of the needle (§ 23). If the variation of the needle at any place is known, by means of the vernier screw the compass circle may be turned through an arc equal to the variation. If the observer stands at the south end of the instrument, the vernier is turned to the right or left according as the variation is west or east. The compass will now give the bearings of the lines with the *true meridian.*

In order to retrace the lines of an old survey, turn the sights in the direction of a known line, and move the vernier until the needle indicates the old bearing. The arc moved over by the vernier will indicate the change of variation since the time of the old survey. If no line is definitely known, the *change of variation* from the time of the old survey will give the arc to be set off.

§ 8. THE SURVEYOR'S TRANSIT.

This instrument is shown on page 17.

The *compass circle* is similar to that of the compass. The *vernier plate* which carries the *telescope* has two *verniers* and moves entirely around the *graduated limb* of the main plate. The axis of the telescope carries a *vertical circle* which measures vertical angles to single minutes by means of a vernier. Under the telescope, and attached to it, is a *spirit level* by means of which horizontal lines may be run, or the difference of level between two stations found. The *cross wires* are two fine fibres of spider's web, or fine platinum wires, which extend across the tube of the telescope at right angles to each other; their intersection determines the *optical axis* or *line of collimation* of the telescope. The transit is levelled by four *levelling screws* which pass through a plate firmly fastened to the ball spindle, and the lower ends of which rest in depressions on the upper side of the tripod plate.

Two recent improvements (introduced by the Messrs. Gurley) enable the surveyor to bring the transit quickly to an approximately level position by the pressure of the hands, after which the levelling screws are used; also, to change the position of the transit without changing the position of the tripod legs, so as to bring the bob exactly over any point.

To level the transit by the levelling screws. Turn the instrument until the spirit levels on the vernier plate are parallel to the vertical planes passing through opposite pairs of levelling screws. Take hold of opposite screw heads with the thumb and fore-finger of each hand, and turn both thumbs in or out as may be necessary to raise the lower side of the parallel plate and lower the other until the desired correction is made.

To use the telescope. Both the eye-piece and the object glass may be moved in and out by a rack-and-pinion movement. The eye-piece must be moved until the cross wires are perfectly distinct; in which case, a slight movement of the eye of the observer, from side to side, will produce no apparent change in the position of the threads upon the object. The object glass must be moved until the object is distinctly visible; and this operation must be repeated whenever the distance of the object is changed.

§ 9. Uses of the Transit.

The transit may be used for all the purposes indicated in § 6, but with much greater precision than the compass. *The principal use, however, of the transit is in measuring horizontal angles by means of the graduated limb and verniers.*

To measure a horizontal angle with the transit. Place the transit over the vertex of the angle; level, and set the limb at zero. Turn the telescope in the direction of one of the sides of the angle, clamp to the spindle; unclamp the main plate, and turn the telescope until it is in the direction of the other side of the angle, and read the angle by the verniers. The object of the two verniers on the vernier plate is to correct any mistake that might arise from the want of exact coincidence in the centres of the verniers and the limb. The correct reading may be obtained by adding to the reading of one vernier the supplement of the reading of the other, and dividing by 2.

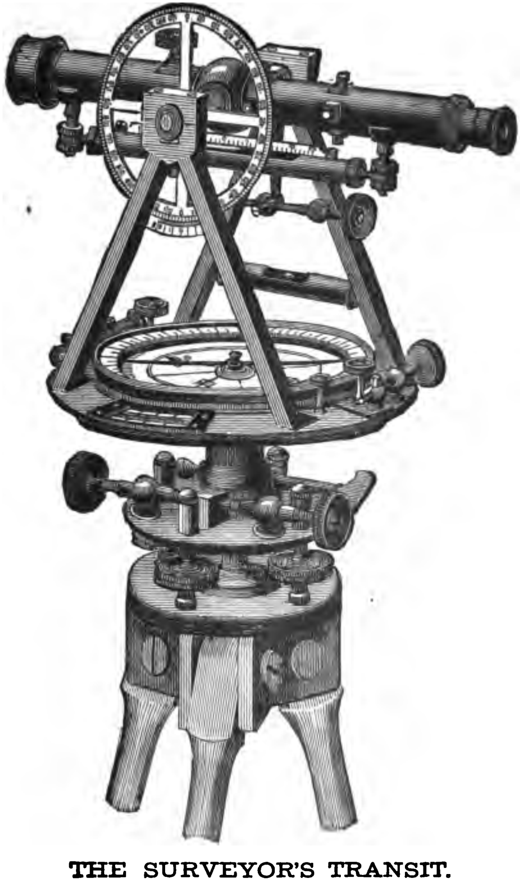
By turning off a right angle by this method, perpendiculars may be constructed with much greater facility than by the chain.

§ 10. The Theodolite.

The telescope of the transit can perform a complete revolution on its axis; whence the name *transit.* The theodolite differs from the transit chiefly in that its telescope cannot be so revolved. It is not much used in this country.

§ 11. The Railroad Compass.

This instrument has all the features of the ordinary compass, and has also a vernier plate and graduated limb for measuring horizontal angles.



THE SURVEYOR'S TRANSIT.

§ 12. PLOTTING.

The principal plotting instruments are a ruler, pencil, straight-line pen, hair-spring dividers, diagonal scale, a right triangle of wood, and a circular protractor. A T-square will also be found convenient.

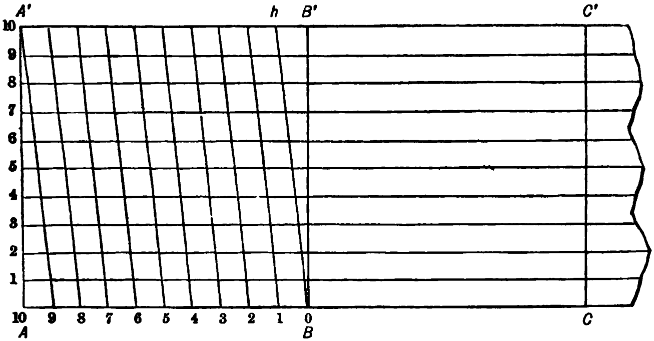


Fig. 13.

The Diagonal Scale. A portion of this scale is shown in Fig. 13. AB is the unit. AB and $A'B'$ are divided into ten equal parts, and B is joined with h, the first division point to the left of B'; the first division point to the left of B is joined with the second to the left of B', etc.

The part of the horizontal line numbered 1 intercepted between BB' and Bh is evidently $\frac{1}{10}$ of $\frac{1}{10} = \frac{1}{100}$ of the unit; the part of the horizontal line numbered 2 intercepted between BB' and Bh is $\frac{2}{100}$ of the unit, etc.

The method of using this scale, in laying off distances, will be made plain by the following example:

Let it be required to lay off the distance 1.43.

Place one foot of the dividers at the intersection of the horizontal line numbered 3 and the diagonal numbered 4, and place the other foot at the intersection of the vertical line numbered 1 (CC') and the horizontal line numbered 3; the distance between the feet of the dividers will be the distance required. For, measuring along the horizontal line numbered 3, from CC' to BB' is 1; from BB' to Bh is .03; and from Bh to the diagonal numbered 4 is .4; and $1 + .03 + .4 = 1.43$.

The Circular Protractor. This instrument (Fig. 14) usually consists of a semi-circular piece of brass or german silver, having its arc divided into degrees and its centre marked.

To lay off an angle with the protractor, place the centre over the vertex of the angle, and make the diameter coincide with the given side of the angle. Mark off the number of degrees in the given angle, and draw a line through this point and the vertex.

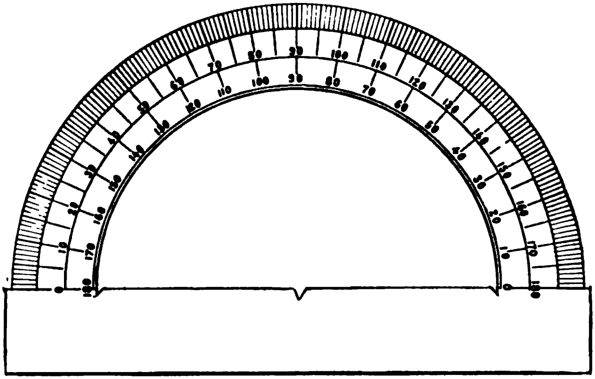


Fig. 14.

Some protractors have an arm which carries a vernier, by which angles may be constructed to single minutes.

To draw through a given point a line parallel to a given line, make one of the sides of a triangle coincide with the given line, and, placing a ruler against one of the other sides, move the triangle along the ruler until the first side passes through the given point; then draw a line along this side.

To draw through a given point a line perpendicular to a given line, make the hypotenuse of a right triangle coincide with the given line, and, placing a ruler against one of the other sides of the triangle, revolve the triangle about the vertex of the right angle as a centre until its other perpendicular side is against the ruler; then move the triangle along the ruler until the hypotenuse passes through the given point, and draw a line along the hypotenuse.

CHAPTER II.

LAND SURVEYING.

§ 13. DEFINITION.

Land Surveying is the art of measuring, laying out, and dividing land, and preparing a plot.

§ 14. DETERMINATION OF AREAS.

The unit of land measure is the

$$\begin{aligned} \text{acre} &= \ \ 10 \text{ square chains} \\ &= \ \ \ 4 \text{ roods} \\ &= 160 \text{ square rods, perches, or poles.} \end{aligned}$$

Areas are referred to the horizontal plane, no allowance being made for inequalities of surface.

For convenience of reference, the following rules for areas are given:

Let A, B, and C be the angles of a triangle, and a, b, and c the opposite sides, respectively; and let $s = \frac{1}{2}(a + b + c)$.

$$\begin{aligned} \text{Area of triangle } ABC &= \sqrt{s(s-a)(s-b)(s-c)} && [\text{A}] \\ &= \tfrac{1}{2} bc \sin A && [\text{B}] \\ &= \tfrac{1}{2} \frac{a^2 \sin B \sin C}{\sin(B+C)} && [\text{C}] \\ &= \tfrac{1}{2} \text{base} \times \text{altitude}. && [\text{D}] \end{aligned}$$

Area of rectangle = base × altitude.

Area of trapezoid = ½ sum of parallel sides × altitude.

PROBLEM 1. **To determine the area of a triangular field.**

Measure the necessary parts with a Gunter's chain, or with a chain and transit, and compute by formula [A], [B], [C], or [D].

PROBLEM 2. To find the area of a field having any number of straight sides.

(*a*) Divide the field into triangles by diagonals; find the area of each triangle and take the sum.

(*b*) Run a diagonal, and perpendiculars from the opposite vertices to this diagonal. The field is thus divided into right triangles, rectangles, and trapezoids, the areas of which may be found and the sum taken.

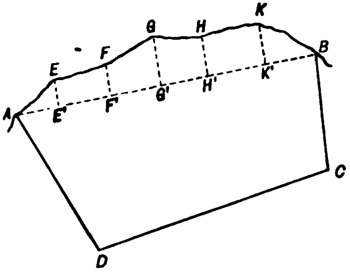


Fig. 15.

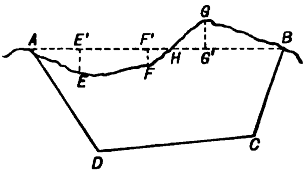


Fig. 16.

PROBLEM 3. To find the area of a field having an irregular boundary line.

(*a*) Let *AGBCD* (Fig. 15) represent a field having a stream *AEFG HKB* as a boundary line. Run the line *AB*. From *E*, *F*, *G*, *H*, and *K*, prominent points on the bank of the stream, let fall perpendiculars *EE'*, *FF'*, etc., upon *AB*. Regarding *AE*, *EF*, etc., as straight lines, the portion of the field cut off by *AB* is divided into right triangles, rectangles, and trapezoids, the necessary elements of which can be measured and the areas computed. The sum of these areas added to the area of *ABCD* will give the area required.

(*b*) When the irregular boundary line crosses the straight line joining its extremities, as in Fig. 16, the areas of *AEFH* and *HGB* may be found separately, as in the preceding case. Then the area required = *ABCD* + *HGB* − *AEFH*.

PROBLEM 4. To determine the area of a field from two interior stations.

Let *ABCD* (Fig. 17) represent a field, and *P* and *P'* two stations within it. Measure *PP'* with great exactness. Measure the angles between *PP'* and the lines from *P* and *P'* to the corners of the field.

In the triangle $PP'D$, PP' and the angles $P'PD$ and $PP'D$ are known; hence, PD may be found. In like manner, PC may be found. Then, in the triangle PDC, PD, PC, and the angle DPC are known; hence, the area of PDC may be computed.

In like manner, the areas of all the triangles about P and P' may be determined.

Area $ABCD = PAD + PDC + PCB + PBA$. Also,

Area $ABCD = P'AD + P'DC + P'CB + P'BA$.

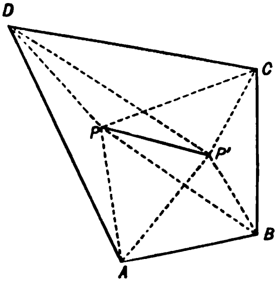


Fig. 17.

PROBLEM 5. To determine the area of a field from two exterior stations.

Let $ABCD$ (Fig. 18) represent the field, and P and P' the stations. Determine the areas of the triangles PAD, PDC, PCB, and PBA, as in the preceding problem.

Area $ABCD = PAD + PDC + PBC - PBA$. Also,

Area $ABCD = P'AD + P'DC + P'BA - P'BC$.

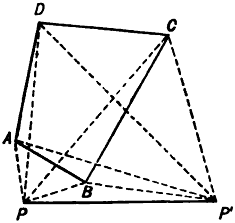


Fig. 18.

EXERCISE I.

1. Required the area of a triangular field whose sides are respectively 13, 14, and 15 chains.

2. Required the area of a triangular field whose sides are respectively 20, 30, and 40 chains.

3. Required the area of a triangular field whose base is 12.60 chains, and altitude 6.40 chains.

4. Required the area of a triangular field which has two sides 4.50 and 3.70 chains, respectively, and the included angle 60°.

5. Required the area of a field in the form of a trapezium, one of whose diagonals is 9 chains, and the two perpendiculars upon this diagonal from the opposite vertices 4.50 and 3.25 chains.

6. Required the area of the field $ABCDEF$ (Fig. 19), if $AE = 9.25$ chains, $FF' = 6.40$ chains, $BE = 13.75$ chains, $DD' = 7$ chains, $DB = 10$ chains, $CC' = 4$ chains, and $AA' = 4.75$ chains.

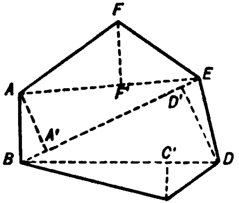


Fig. 19.

7. Required the area of the field $ABCDEF$ (Fig. 20), if $AF' = 4$ chains, $FF' = 6$ chains, $EE' = 6.50$ chains, $AE' = 9$ chains, $AD = 14$ chains, $AC' = 10$ chains, $AB' = 6.50$ chains, $BB' = 7$ chains, $CC' = 6.75$ chains.

8. Required the area of the field $AGBCD$ (Fig. 15), if the diagonal $AC = 5$, BB' (the perpendicular from B to AC) $= 1$, DD' (the perpendicular from D to AC) $= 1.60$, $EE' = 0.25$, $FF' = 0.25$, $GG' = 0.60$, $HH' = 0.52$, $KK' = 0.54$, $AE' = 0.2$, $E'F' = 0.50$, $F'G' = 0.45$, $G'H' = 0.45$, $H'K' = 0.60$, and $K'B = 0.40$.

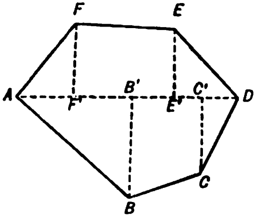


Fig. 20.

9. Required the area of the field $AGBCD$ (Fig. 16), if $AD = 3$, $AC = 5$, $AB = 6$, angle $DAC = 45°$, angle $BAC = 30°$, $AE' = 0.75$, $AF' = 2.25$, $AH = 2.53$, $AG' = 3.15$, $EE' = 0.60$, $FF' = 0.40$, and $GG' = 0.75$.

10. Determine the area of the field $ABCD$ from two interior stations, P and P', if $PP' = 1.50$ chains,

angle $PP'C = 89°\,35'$, angle $P'PB = 3°\,35'$,
$PP'B = 185°\,30'$, $P'PA = 113°\,45'$,
$PP'A = 309°\,15'$, $P'PD = 165°\,40'$,
$PP'D = 349°\,45'$, $P'PC = 303°\,15'$.

11. Determine the area of the field $ABCD$ from two exterior stations P and P', if $PP' = 1.50$ chains,

angle $P'PB = 41°\,10'$, angle $PP'D = 66°\,45'$,
$P'PA = 55°\,45'$, $PP'C = 95°\,40'$,
$P'PC = 77°\,20'$, $PP'B = 132°\,15'$,
$P'PD = 104°\,45'$, $PP'A = 103°\;0'$.

RECTANGULAR SURVEYING.

§ 15. DEFINITIONS.

An **East and West Line** is a line perpendicular to the magnetic meridian.

The **Latitude** of a line is the distance between the east and west lines through its extremities.

The **Departure** of a line is the distance between the meridians through its extremities.

NOTE. When a line extends north of the initial point the latitude is called a northing; when it extends south, a southing; when it extends east the departure is called an easting; when it extends west, a westing.

The **Meridian Distance** of a point is its distance from a meridian.

Let AB (Fig. 21) represent a line, and NAS the magnetic meridian. Let BB' be perpendicular to NS.

The bearing of the line AB is the angle BAB'.

The latitude of the line AB is AB'.

The departure of the line AB is BB'.

The meridian distance of the point B is BB'.

In the right triangle ABB',

$$AB' = AB \times \cos BAB',$$

and $$BB' = AB \times \sin BAB'.$$

Hence, *latitude* = *distance* × *cos of bearing*,

and *departure* = *distance* × *sin of bearing*.

The latitudes and departures corresponding to any distance and bearing may be found from the above formulas by means of a table of natural sines and cosines, or from "The Traverse Table." *

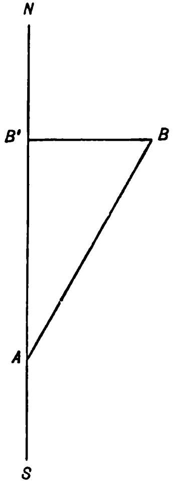


Fig. 21.

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* See Table VII. of Wentworth & Hill's Five-Place Logarithmic and Trigonometric Tables.

§ 16. Field Notes, Computation, and Plotting.

The field notes are kept in a book provided for the purpose. The page is ruled in three columns, in the first of which is written the number of the station; in the second, the bearing of the side; and in the third, the length of the side.

Example 1. To survey the field *ABCD* (Fig. 22).

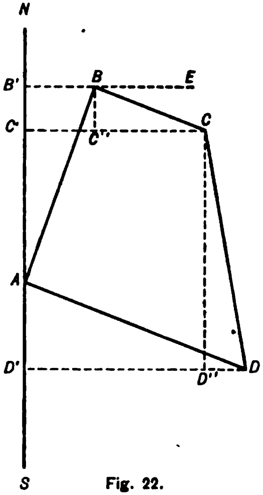


Fig. 22.

Field Notes.

1	N. 20° E.	8.66
2	S. 70° E.	5.00
3	S. 10° E.	10.00
4	N. 70° W.	10.00

(*a*) To obtain the field notes.

Place the compass at *A*, the first station, and take the bearing of *AB* (§ 6); suppose it to be N. 20° E. Write the result in the second column of the field notes opposite the number of the station. Measure $AB = 8.66$ chains, and write the result in the third column of the field notes.

Place the compass at *B*, and, after testing the bearing of *AB* (§ 6), take the bearing of *BC*, measure *BC*, and write the results in the field notes; and so continue until the bearing and length of each side have been recorded.

(*b*) To compute the area.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.
Side.	*Bearing.*	*Dist.*	*N.*	*S.*	*E.*	*W.*	*M.D.*	*D.M.D.*	*N.A.*	*S.A.*
AB	N. 20° E.	8.66	*AB'* 8.14	....	*BB'* 2.96	...	*BB'* 2.96	*BB'* 2.96	2*ABB'* 24.0944	
BC	S. 70° E.	5.00	....	*B'C'* 1.71	*CC''* 4.70	...	*CC'* 7.66	*BB'*+*CC'* 10.62	....	2*C'CBB'* 18.1602
CD	S. 10° E.	10.00	....	*C'D'* 9.85	*DD''* 1.74	...	*DD'* 9.40	*CC'*+*DD'* 17.06	....	2*D'DCC'* 168.0410
DA	N. 70° W.	10.00	*D'A* 3.42	....	....	*DD'* 9.40	0	*DD'* 9.40	2*ADD'* 32.1480	
									56.2424	186.2012

The survey may begin at any corner of the field; but in computing the area, the field notes should be arranged so that the most eastern or most western station will stand first. For the sake of uniformity, we shall always begin with the most *western* station, and keep the field on the *right* in passing around it.

$$\begin{array}{r|l} & 186.2012 \\ & \underline{56.2424} \\ 2 & \underline{129.9588} \\ 10 & \underline{64.98} \text{ sq. chains.} \\ & 6.498 \text{ acres.} \end{array}$$

The field notes occupy the first three of the eleven columns in the above tablet. Columns IV., V., VI., and VII. contain the latitudes and departures corresponding to the sides, and taken from the Traverse Table. The lines represented by these numbers are indicated immediately above each number. Column VIII. contains the meridian distances of the points B, C, D, and A, taken in order. Column IX. contains the double meridian distances. Their composition is indicated by the letters immediately above the numbers. Column X. contains the products of the double meridian distances by the northings in the same line. The first number,

$$24.0944 = 2.96 \times 8.14 = BB' \times AB' = 2 \text{ area of the triangle } ABB';$$
$$32.1480 = 9.40 \times 3.42 = DD' \times AD' = 2 \text{ area of the triangle } ADD'.$$

Column XI. contains the products of the double meridian distances by the southings in the same line. The first number,

$$\begin{aligned} 18.1602 = 10.62 \times 1.71 &= (BB' + CC') \times B'C' \\ &= 2 \text{ area of the trapezoid } C'CBB'; \\ 168.0410 = 17.06 \times 9.85 &= (CC' + DD') \times D'C' \\ &= 2 \text{ area of the trapezoid } D'DCC'. \end{aligned}$$

The sum of the north areas in column X.

$$= 56.2424 = 2(ABB' + ADD').$$

The sum of the south areas in column XI.

$$= 186.2012 = 2(C'CBB' + D'DCC').$$

But $(C'CBB' + D'DCC') - (ABB' + ADD') = ABCD.$

Hence, $2(C'CBB' + D'DCC') - 2(ABB' + ADD') = 2ABCD;$

that is, $186.2012 - 56.2424 = 129.9588 = 2ABCD.$

Hence, area $ABCD = \frac{1}{2}$ of $129.9588 = 64.9794$ sq. ch. $= 6.498$ acres.

(*c*) To make the plot.

The plot or map may be drawn to any desired scale. If a line one inch in length in the plot represents a line one chain in length, the plot is said to be drawn to a scale of one chain to an inch. In this case the plot (Fig. 22) is drawn to a scale of eight chains to an inch.

Draw the line NAS to represent the magnetic meridian, and lay off the first northing $AB' = 8.14$ (§ 12). Draw the indefinite line $B'E$ perpendicular to NS and lay off $B'B$, the first easting $= 2.96$. Join A and B; then the line AB will represent the first side of the field. Through B draw BC'' perpendicular to BB', and make $BC'' = 1.71$, the first southing. Through C'' draw $C''C$ perpendicular to BC'', and equal to 4.70, the second easting. Join B and C. The line BC will represent the second side of the field.

Proceed in like manner until the field is completely represented. The extremity of the last line $D'A$, measured from D', should fall at A. This will be a test of the accuracy of the plot.

By drawing the diagonal AC, and letting fall upon it perpendiculars from B and D, the quadrilateral $ABCD$ is divided into two triangles, the bases and altitudes of which may be measured and the area computed approximately.

Other methods of plotting will suggest themselves, but the method just explained is one of the best.

Balancing the Work.

In the survey, we pass entirely around the field; hence, we move just as far north as south. Therefore, the sum of the northings should equal the sum of the southings. In like manner, the sum of the eastings should equal the sum of the westings. In this way the accuracy of the field work may be tested.

In Example 1, the sum of the northings is equal to the sum of the southings, being 11.56 in each case; and the sum of the eastings is equal to the sum of the westings, being 9.40 in each case. Hence, the work balances.

In actual practice the work seldom balances. When it does not balance, corrections are generally applied to the latitudes and departures, by the following rules:

The perimeter of the field : any one side
: : total error in latitude : correction;
: : total error in departure : correction.

If special difficulty has been experienced in taking a particular bearing, or in measuring a particular line, the corrections should be applied to the corresponding latitudes and departures.

The amount of error allowable varies in the practice of different surveyors, and according to the nature of the ground. An error of 1 link in 8 chains would not be considered too great on smooth, level ground; while, on rough ground, an error of 1 link in 2 or 3 chains might be allowed. If the error is considerable, the field measurements should be repeated.

EXAMPLE 2. Let it be required to survey the field *ABCDEF* (Fig. 23).

FIELD NOTES.

1	N. 73° 30′ W.	5.00
2	S. 16° 30′ W.	5.00
3	N. 28° 30′ W.	7.07
4	N. 20° 00′ E.	11.18
5	S. 43° 30′ E.	5.00
6	S. 13° 30′ E.	10.00

Side.	*Bearing.*	*Dist.*	*N.*	*S.*	*E.*	*W.*	*N'.*	*S'.*	*E'.*	*W'.*	*M. D.*	*D. M. D.*	*N. A.*	*S. A.*
AB	N.20° 00' E.	11.18	10.51	...	3.82	...	*AB'* 10.51	...	*B'B* 3.82	....	*B'B* 3.82	*B'B* 3.82	2 *ABB'* 40.1482	....
BC	S.43° 30' E.	5.00	...	3.63	3.44	...	....	*B'C'* 3.63	*C''C* 3.44	....	*C'C* 7.26	*B'B* + *C'C* 11.08	....	2 *C'CBB'* 40.2204
CD	S.13° 30' E.	10.00	...	9.72	2.33	...	....	*C'D'* 9.72	*D''D* 2.33	....	*D'D* 9.59	*C'C* + *D'D* 16.85	....	2*D'DCC'* 163.7820
DE	N.73° 30' W.	5.00	1.42	...	...	4.79	*D'E'* 1.42	...	....	*DD'''* 4.80	*E'E* 4.79	*D'D* + *E'E* 14.38	2*D'DEE'* 20.4196	....
EF	S.16° 30' W.	5.00	...	4.79	...	1.42	....	*E'F'* 4.79	....	*F''F* 1.42	*F'F* 3.37	*E'E* + *F'F* 8.16	....	2*F'FEE'* 39.0864
FA	N.28° 30' W.	7.07	6.21	...	...	3.37	*F'A* 6.21	...	....	*F F* 3.37	0	*F'F* 3.37	2 *AFF'* 20.9277	....
			18.14	18.14	9.59	9.58							81.4955	243.0888

243.0888
81.4955
2 | 161.5933
10 | 80.7967
8.0797 acres.

EXPLANATION. The first station in the field notes is *D*, but we rearrange the numbers in the tablet so that *A* stands first. The northings and southings balance, but the eastings exceed the westings by 1 link. We apply the correction to the westing 4.79 (the distance *DE* being in doubt), making it 4.80, and rewrite all the latitudes and departures in the next four columns, incorporating the correction. In practice, the corrected numbers are written in red ink.

The remainder of the computation does not require explanation.

It will be seen that this method of computing areas is perfectly general.

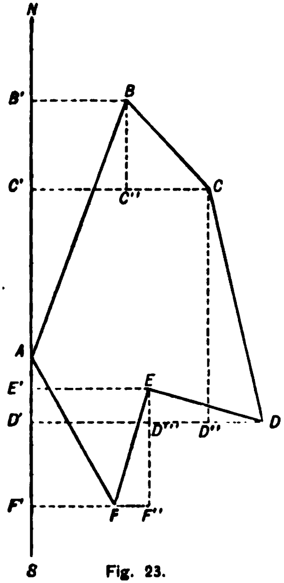


Fig. 23.

§ 17. SUPPLYING OMISSIONS.

If, for any reason, the bearing and length of any side do not appear in the field notes, the latitude and departure of this side may be found in the following manner:

Find the latitudes and departures of the other sides as usual. The difference between the northings and southings will give the northing or southing of the unknown side, and the difference between the eastings and westings will give the easting or westing of the unknown side.

If the length and bearing of the unknown side are desired, they may be found by solving the right triangle, whose sides are the latitude and departure found by the method just explained, and whose hypotenuse is the length required.

§ 18. IRREGULAR BOUNDARIES.

If a field have irregular boundaries, its area may be found by offsets, as explained in § 14, Prob. 3.

§ 19. OBSTRUCTIONS.

If the end of a line be not visible from its beginning, or if the line be inaccessible, its length and bearing may be found as follows:

1. By means of a random line (§ 4, 3).
2. When it is impossible to run a random line, which is frequently the case on account of the extent of the obstruction, the following method may be used:

Let AB (Fig. 24) represent an inaccessible line whose extremities A and B only are known, and B invisible from A.

Set flag-staffs at convenient points, C and D. Find the bearings and lengths of AC, CD, and DB, and then proceed to find the latitude and departure of AB, as in § 17.

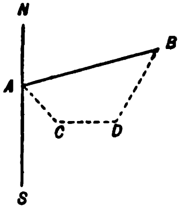


Fig. 24.

EXAMPLE. Suppose that we have the following notes (see Fig. 24):

Side.	Bearing.	Dist.	N.	S.	E.	W.
AC	S. 45° E.	3.00		2.12	2.12	
CD	E.	3.50			3.50	
DB	N. 30° E.	4.83	4.18		2.42	
			4.18	2.12	8.04	0

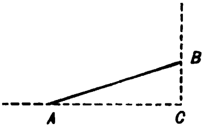


Fig. 25.

$$\begin{array}{r} 4.18 \\ \underline{2.12} \\ 2.06 \end{array}$$

The northing of AB is 2.06, and the easting, 8.04; which numbers may be entered in the tablet in the columns N. and E., opposite the side AB.

If the bearing and length of AB are required, construct the right triangle ABC (Fig. 25), making $AC = 8.04$ and $BC = 2.06$.

$$\tan BAC = \frac{BC}{AC} = \frac{2.06}{8.04} = 0.256.$$

Hence, the angle $BAC = 14° 22'$.

Also, $$AB = \sqrt{\overline{AC}^2 + \overline{BC}^2} = \sqrt{8.04^2 + 2.06^2} = 8.29.$$

Therefore, the bearing and length of AB are N. 75° 38' E. 8.29.

EXERCISE II.

NOTE. In examples 5 and 6 detours were made on account of inaccessible sides (§ 19, 2). The notes of the detours are written in braces.

1.

Sta.	Bearings.	Dist.
1	S. 75° E.	6.00
2	S. 15° E.	4.00
3	S. 75° W.	6.93
4	N. 45° E.	5.00
5	N. 45° W.	5.19½

2.

Sta.	Bearings.	Dist.
1	N. 45° E.	10.00
2	S. 75° E.	11.55
3	S. 15° W.	18.21
4	N. 45° W.	19.11

3.

Sta.	Bearings.	Dist.
1	N. 15° E.	3.00
2	N. 75° E.	6.00
3	S. 15° W.	6.00
4	N. 75° W.	5.20

4.

Sta.	Bearings.	Dist.
1	N. 89°45′E.	4.94
2	S. 7°00′W.	2.30
3	S. 28°00′E.	1.52
4	S. 0°45′E.	2.57
5	N. 84°45′W.	5.11
6	N. 2°30′W.	5.79

5.

Sta.	Bearings.	Dist.
1	S. 2°15′E.	9.68
2	N. 51°45′W.	2.39
	S. 85°00′W.	6.47
	S. 55°10′W.	1.62
3	N. 3°45′E.	6.39
4	S. 66°45′E.	1.70
5	N. 15°00′E.	4.98
6	S. 82°45′E.	6.03

6.

Sta.	Bearings.	Dist.
1	S. 81°20′W.	4.28
	N. 76°30′W.	2.67
2	N. 5°00′E.	8.68
3	S. 87°30′E.	5.54
4	S. 7°00′E.	1.79
	S. 27°00′E.	1.94
	S. 10°30′E.	5.35
	N. 76°45′W.	1.70

7.

Sta.	Bearings.	Dist.
1	N. 6°15′W.	6.31
2	S. 81°50′W.	4.06
3	S. 5°00′E.	5.86
4	N. 88°30′E.	4.12

8.

Sta.	Bearings.	Dist.
1	N. 5°30′W.	6.08
2	S. 82°30′W.	6.51
3	S. 3°00′E.	5.33
4	E.	6.72

9.

Sta.	Bearings.	Dist.
1	N. 20°00′E.	4.62½
2	N. 73°00′E.	4.16½
3	S. 45°15′E.	6.18½
4	S. 38°30′W.	8.00
5	Wanting.	Wanting.

10.

Sta.	Bearings.	Dist.
1	S. 3°00′E.	4.23
2	S. 86°45′W.	4.78
3	S. 37°00′W.	2.00
4	N. 81°00′W.	7.45
5	N. 61°00′W.	2.17
6	N. 32°00′E.	8.68
7	S. 75°50′E.	6.38
8	S. 14°45′W.	0.98
9	S. 79°15′E.	4.52

§ 20. MODIFICATION OF THE RECTANGULAR METHOD.

The area of a field may be found by a modification of the rectangular method, if its sides and interior angles are known.

Let A, B, C, D, represent the interior angles of the field $ABCD$ (Fig. 26). Let the side AB determine the direction of reference.

The bearing of AB, with reference to AB, is 0°.

The bearing of BC, with reference to AB, is the angle $b = 180° - B$.

The bearing of CD, with reference to AB, is the angle $c = C - b$.

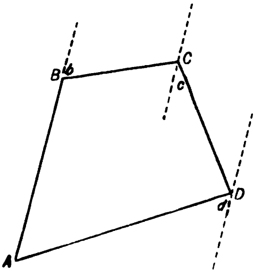


Fig. 26.

The bearing of DA, with reference to AB, is the angle $d = A$.

The area may now be computed by the rectangular method, regarding AB as the magnetic meridian.

As the interior angles may be measured with considerable accuracy by the transit, the latitudes and departures should be obtained by using a table of natural sines and cosines.

EXERCISE III.

1. Find the area of the field $ABCD$, in which the angle $A = 120°$, $B = 60°$, $C = 150°$, and $D = 30°$; and the side $AB = 4$ chains, $BC = 4$ chains, $CD = 6.928$ chains, and $DA = 8$ chains.

2. Find the area of the farm $ABCDE$, in which the angle $A = 106° 19'$, $B = 99° 40'$, $C = 120° 20'$, $D = 86° 8'$, and $E = 127° 33'$; and the side $AB = 79.86$ rods, $BC = 121.13$ rods, $CD = 90$ rods, $DE = 100.65$ rods, and $EA = 100$ rods.

§ 21. GENERAL REMARKS ON DETERMINING AREAS.

Operations depending upon the reading of the magnetic needle must lack accuracy. Hence, when great accuracy is required (which is seldom the case in land surveying), the rectangular method (§§ 16–19) cannot be employed.

The best results are obtained by the methods explained in §§ 14 and 20, the horizontal angles being measured with the transit, and great care exercised in measuring the lines.

§ 22. THE VARIATION OF THE NEEDLE.

The **Magnetic Declination,** or variation of the needle, at any place, is the angle which the magnetic meridian makes with the true meridian, or north and south line. The variation is said to be east or west, according as the north end of the needle lies east or west of the true meridian. Western variation is indicated by the sign +, and eastern variation by the sign −.

Irregular Variations are sudden deflections of the needle, which occur without apparent cause. They are sometimes accompanied by auroral displays and thunder storms, and are most frequent in years of greatest sun-spot activity.

Solar-Diurnal Variation. North of the equator, the north end of the needle moves to the west, from 8 A.M. to 1.30 P.M., about 6′ in winter and 11′ in summer, and then returns gradually to its normal position.

Secular Variation is a change in the same direction for about a century and a half; then in the opposite direction for about the same time.

The line of no variation, or the **Agonic Line,** is a line joining those places at which the magnetic meridian coincides with the true meridian. In the United States, this line at present (1882) passes through Michigan, Ohio, the Virginias, and the Carolinas. It is moving gradually westward, so that the variation is increasing at places east of this line, and decreasing at places west of this line. East of this line the variation is westerly, and west of this line the variation is easterly.

The table on page 35, which has been prepared by permission from the report for 1879 of the United States Coast and Geodetic Survey, shows the magnetic variation at different places in the United States and Canada for several years; also, the annual change for 1880.

Place.	Latitude.	Longitude.	Variation.							Annual Change.
			1800	1820	1840	1860	1870	1880	1885	1880
	Deg. Min.	Deg. Min.	Degrees.	Degrees.	Degrees.	Degrees.	Degrees.	Degrees.	Degrees.	Minutes.
Portland, Me.	43 38.8	70 16.6	8.90	10.00	11.23	12.35	12.80	13.13	13.25	1.6
Portsmouth, N.H.	43 04.2	70 42.5	7.60	8.30	9.59	11.16	11.92	12.60	12.90	3.7
Boston, Mass.	42 21.5	71 03.8	6.98	7.88	9.11	10.43	11.03	11.56	11.78	2.9
Cambridge, Mass.	42 22.9	71 07.7	7.10	8.00	9.33	10.67	11.21	11.63	11.78	2.1
Quebec, Canada	46 48.4	71 14.5	12.15	12.23	13.70	15.95	16.82	17.31	17.40	1.6
Providence, R.I.	41 49.5	71 24.1	6.37	6.73	8.31	9.65	10.21	10.90	....	....
Hartford, Conn.	41 45.9	72 40.4	5.16	5.46	6.24	7.36	7.99	8.62	8.92	3.7
New Haven, Conn.	41 18.5	72 55.7	4.67	4.98	5.95	7.35	8.10	8.84	9.19	4.3
Burlington, Vt.	44 28.2	73 12.3	7.30	7.49	8.95	10.26	10.98	11.91	12.41	6.0
Albany, N.Y.	42 39.2	73 45.8	....	5.81	7.00	8.49	9.23	9.90	10.19	3.7
New York, N.Y.	40 42.7	74 00.4	4.25	4.44	5.56	6.93	7.40	7.81	8.03	2.5
Philadelphia, Pa.	39 56.9	75 09.0	1.90	2.54	3.63	4.76	5.45	6.26	6.70	4.9
Baltimore, Md.	39 17.8	76 37.0	0.64	0.88	1.70	2.90	3.55	4.17	4.47	3.6
Harrisburg, Pa.	40 15.9	76 52.9	0.04	0.83	2.17	3.71	4.43	5.05	5.30	3.3
Washington, D.C.	38 53.3	77 00.6	0.00	0.40	1.29	2.45	3.04	3.61	3.87	3.2
Buffalo, N.Y.	42 52.8	78 53.5	0.03	0.22	1.16	2.64	3.48	4.32	4.73	5.0
Toronto, Can.	43 39.4	79 23.4	....	....	1.32	2.17	2.66	3.62	3.88	4.5
Charleston, S.C.	32 46.6	79 55.8	−4.70	−3.97	−2.84	−1.56	−0.95	−0.41	−0.17	3.0
Erie, Pa.	42 07.8	80 05.4	−0.33	−0.37	0.34	1.60	2.32	3.04	3.39	4.2
Savannah, Ga.	32 04.9	81 05.5	....	−4.80	−4.14	−3.08	−2.48	−1.89	−1.60	3.5
Cleveland, O.	41 30.3	81 42.0	−1.90	−1.40	−0.61	0.38	0.87	1.31	1.51	2.5
Key West, Fla.	24 33.5	81 48.5	....	−6.70	−6.06	−4.88	−4.18	−3.47	−3.12	4.2
Detroit, Mich.	42 20.0	83 03.0	−3.18	−2.90	−2.09	−0.99	−0.41	0.13	0.37	3.0
Mobile, Ala.	30 41.4	88 02.5	−5.80	−6.73	−7.09	−6.77	−6.38	−5.86	−5.56	3.4
New Orleans, La.	29 57.2	90 03.9	−7.52	−8.11	−8.10	−7.52	−7.05	−6.49	−6.18	3.5
St. Louis, Mo.	38 38.0	90 12.2	....	−9.20	−8.40	−7.39	−6.83	−6.28	−6.00	3.2
Omaha, Neb.	41 15.7	95 56.5	....	....	....	....	....	10.10	....	2.6
San Diego, Cal.	32 42.1	117 14.3	−11.10	−11.60	−12.20	−12.88	−13.20	−13.50	−13.64	−1.7
San Francisco, Cal.	37 47.5	122 27.2	−13.30	−14.42	−15.40	−16.13	−16.37	−16.52	−16.56	−0.5

§ 23. To Establish a True Meridian.

This may be done as follows:

1. *By means of Burt's Solar Compass* (§ 25).
2. *By observations of Polaris.*

The North Star or Polaris revolves about the pole at present at the distance of about 1⅓°; hence, it is on the meridian twice in 23 h. 56 m. (a siderial day), once above the pole called the upper culmination, and 11 h. 58 m. later below the pole called the lower culmination. It attains its greatest eastern or western elongation, or greatest distance from the meridian, 5 h. 59 m. after the culmination.

The following table gives the time of the upper culmination of Polaris for 1882. The time is growing later at the rate of about one minute in three years.

Month.	First Day.	Eleventh Day.	Twenty-first Day.
	H. M.	H. M.	H. M.
January . .	6 32 P.M.	5 53 P.M.	5 13 P.M.
February .	4 30 P.M.	3 50 P.M.	3 11 P.M.
March . . .	2 35 P.M.	1 56 P.M.	1 17 P.M.
April. . . .	12 29 P.M.	11 50 A.M.	11 11 A.M.
May	10 31 A.M.	9 52 A.M.	9 13 A.M.
June	8 30 A.M.	7 51 A.M.	7 12 A.M.
July	6 32 A.M.	5 53 A.M.	5 14 A.M.
August. . .	4 31 A.M.	3 52 A.M.	3 13 A.M.
September .	2 29 A.M.	1 50 A.M.	1 11 A.M.
October . .	12 32 A.M.	11 52 P.M.	11 13 P.M.
November .	10 30 P.M.	9 50 P.M.	9 11 P.M.
December .	8 32 P.M.	7 52 P.M.	7 13 P.M.

The time of the upper culmination of Polaris may be found by means of the star Alioth, which is the star in the handle of the Dipper (in the constellation of the Great Bear) next to the four which form the bowl. It crosses the meridian about the same time as Polaris. Suspend from a height of about 20 feet a plumb-line, placing the bob in a pail of water to lessen its vibrations. About 15 feet south of the plumb-line, upon a horizontal board firmly supported at a convenient height, place a compass sight fastened to a board a few inches square. At night, when Alioth by estimation approaches the meridian, place the compass sight in line with Polaris and the plumb-line, and move it so as to keep it in this line until the plumb-line also falls on Alioth (Fig 27). Note the time; then twenty-two minutes later Polaris will be on the meridian.

If the lower culmination takes place at night, the time may be found in a similar manner.

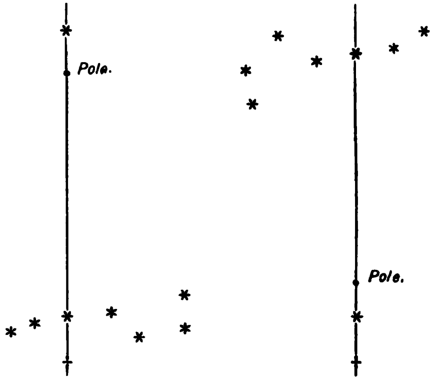


Fig. 27

Instead of the compass sight, any upright with a small opening or slit may be used.

The plumb-line may be made visible by a light held near it.

(*a*) **To locate the true meridian by the position of Polaris at its culmination.**

1. *By using the apparatus described in finding the time of culmination.* At the time of culmination bring Polaris, the plumb-line, and the compass sight into line. The compass sight and the plumb-bob will give two points in the true meridian.

2. *By means of the transit.* Bring the telescope to bear on Polaris at the time of culmination, holding a light near to make the wires visible, if the observation is made at night. The telescope will then lie in the plane of the meridian, which may be marked by bringing the telescope to a horizontal position.

(*b*) To locate the meridian by the position of Polaris at greatest elongation.

The **Azimuth** of a star is the angle which the meridian plane makes with a vertical plane passing through the star and the zenith of observer.

Let Z (Fig. 28) represent the zenith of the place, P the pole, and N Polaris at its greatest elongation. Let ZP, ZN, and PN be arcs of great circles, and let N be a right angle.

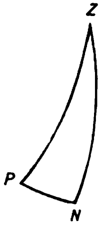


Fig. 28.

$$\sin PN = \cos(90^\circ - ZP)\cos(90^\circ - Z).$$

[Spher. Trig. § 47.]

But $ZP =$ the complement of the latitude.

Hence, $90^\circ - ZP =$ the latitude of the place.

Hence, $\sin PN = \cos \text{latitude} \times \sin Z.$

Hence, $$\sin Z = \frac{\sin PN}{\cos \text{latitude}}.$$

Hence, Z (the azimuth of Polaris) can be found if the latitude of the place and the greatest elongation of Polaris (PN) are known.

The following table gives the latter element Jan. 1, 1882–93.

GREATEST ELONGATION OF POLARIS.

1882	1° 19′ 12.5″	1886	1° 17′ 56.6″	1890	1° 16′ 40.7″
1883	1° 18′ 53.5″	1887	1° 17′ 37.6″	1891	1° 16′ 21.8″
1884	1° 18′ 34.5″	1888	1° 17′ 18.6″	1892	1° 16′ 3″
1885	1° 18′ 15.5″	1889	1° 16′ 59.7″	1893	1° 15′ 44.1″

The greatest elongation of Polaris, or the polar distance, is given in the Nautical Almanac. The table gives this element for Jan. 1. It may be found for other dates by interpolation.

To obtain a line in the direction of Polaris at greatest elongation.

1. *By using the apparatus for finding the time of culmination.* A few minutes before the time of greatest elongation (5 h. 59 m. after culmination), place the compass sight in line with the plumb-line and Polaris, and keep it in line with these until the star begins to recede. At this moment the sight and plumb-line are in the required line.

2. *By means of the transit.* A few minutes before the time of greatest elongation, bring the telescope to bear on the star, and follow it, keeping the vertical wire over the star until it begins to recede. The telescope will then be in the required line.

To establish the meridian. Having the transit sighted in the direction of the line just found, turn it through an angle equal to the azimuth in the proper direction.

§ 24. DIVIDING LAND.

A few problems in the Division of Land are solved in this section, for the purpose of acquainting the student with some of the methods employed. The surveyor must, for the most part, depend on his general knowledge of Geometry and Trigonometry, and his own ingenuity, for solutions of the problems that arise in practice.

PROBLEM 1. **To divide a triangular field into two parts having a given ratio, by a line through a given vertex.**

Let ABC (Fig. 29) be the triangle, and A the given vertex.

Divide BC at D, so that $\frac{BD}{DC}$ equals the given ratio, and join A and D. ABD and ADC will be the parts required; for

$$ABD : ADC :: BD : DC.$$

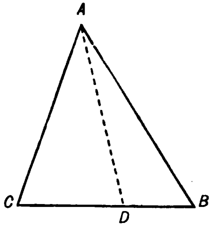


Fig. 29.

PROBLEM 2. **To cut off from a triangular field a given area, by a line parallel to the base.**

Let ABC (Fig. 30) be the triangle, and let DE be the division line required.

$$\sqrt{ABC} : \sqrt{ADE} :: AB : AD.$$

$$\therefore AD = AB\sqrt{\frac{ADE}{ABC}}.$$

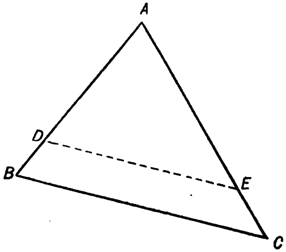


Fig 30.

PROBLEM 3. To divide any field into two parts having a given ratio, by a line through a given point in the perimeter.

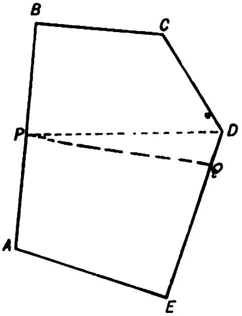


Fig 31.

Let $ABCDE$ (Fig. 31) represent the field, P the given point, and PQ the required division line.

The areas of the whole field and of the required parts having been determined, run the line PD from P to a corner D, dividing the field, as near as possible, as required. Determine the area $PBCD$.

The triangle PDQ represents the part which must be added to $PBCD$ to make the required division.

$$\text{Area } PDQ = \tfrac{1}{2} \times PD \times DQ \times \sin PDQ.$$

$$\text{Hence, } DQ = \frac{2 \text{ area } PDQ}{PD \times \sin PDQ}.$$

NOTE. $DQ = \dfrac{2 \text{ area } PDQ}{\text{perpendicular from } P \text{ on } DE}$. This perpendicular from P on DE may be run and measured directly.

PROBLEM 4. To divide a field into a given number of parts, so that access to a pond of water is given to each.

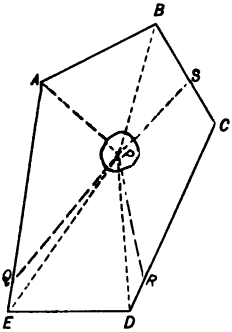


Fig. 32.

Let $ABCDE$ (Fig. 32) represent the field, and P the pond. Let it be required to divide the field into four parts. Find the area of the field and of each part.

Let AP be one division line. Run PE, and find the area APE. Take the difference between APE and the area of one of the required parts; this will give the area of the triangle PQE, from which QE may be found, as in Problem 3. Join P and Q; PAQ will be one of the required parts. In like manner, PQR and PAS are determined; whence, PSR must be the fourth part required.

Exercise IV.

1. From the square $ABCD$, containing 6 A. 1 R. 24 P., part off 3 A. by a line EF parallel to AB.

2. From the rectangle $ABCD$, containing 8 A. 1 R. 24 P., part off 2 A. 1 R. 32 P. by a line EF parallel to $AD = 7$ ch. Then, from the remainder of the rectangle, part off 2 A. 3 R. 25 P., by a line GH parallel to EB.

3. Part off 6 A. 3 R. 12 P. from a rectangle $ABCD$, containing 15 A., by a line EF parallel to AB; AD being 10 ch.

4. From a square $ABCD$, whose side is 9 ch., part off a triangle which shall contain 2 A. 1 R. 36 P., by a line BE drawn from B to the side AD.

5. From $ABCD$, representing a rectangle, whose length is 12.65 ch., and breadth 7.58 ch., part off a trapezoid which shall contain 7 A. 3 R. 24 P., by a line BE drawn from B to the side DC.

6. In the triangle ABC, $AB = 12$ ch., $AC = 10$ ch., and $BC = 8$ ch.; part off 1 A. 2 R. 16 P., by the line DE parallel to AB.

7. In the triangle ABC, $AB = 26$ ch., $AC = 20$ ch., and $BC = 16$ ch.; part off 6 A. 1 R. 24 P., by the line DE parallel to AB.

8. It is required to divide the triangular field ABC among three persons whose claims are as the numbers 2, 3, and 5, so that they may all have the use of a watering-place at C; $AB = 10$ ch., $AC = 6.85$ ch., and $CB = 6.10$ ch.

9. Divide the five-sided field $ABCHE$ among three persons, X, Y, and Z, in proportion to their claims, X paying \$500, Y paying \$750, and Z paying \$1000, so that each may have the use of an interior pond at P, the quality of the land being equal throughout. Given $AB = 8.64$ ch., $BC = 8.27$ ch., $CH = 8.06$ ch., $HE = 6.82$ ch., and $EA = 9.90$ ch. The perpendicular PD upon $AB = 5.60$ ch., PD' upon $BC = 6.08$ ch., PD'' upon $CH = 4.80$ ch., PD''' upon $HE = 5.44$ ch., and PD'''' upon $EA = 5.40$ ch. Assume PH as the divisional

fence between X 's and Z 's shares; it is required to determine the position of the fences *PM* and *PN* between X 's and Y 's shares and Y 's and Z 's shares, respectively.

10. Divide the triangular field *ABC*, whose sides *AB*, *AC*, and *BC* are 15, 12, and 10 ch., respectively, into three equal parts, by fences *EG* and *DF* parallel to *BC*.

11. Divide the triangular field *ABC*, whose sides *AB*, *BC*, and *AC* are 22, 17, and 15 ch., respectively, among three persons, A, B, and C, by fences parallel to the base *AB*, so that A may have 3 A., B, 4 A., and C, the remainder.

25. UNITED STATES PUBLIC LANDS.

Burt's Solar Compass.

This instrument, which is exhibited on the following page, may be used for most of the purposes of a compass or transit. Its most important use, however, is to run north and south lines in laying out the public lands.

A full description of the solar compass, with its principles, adjustments, and uses, forms the subject of a considerable volume, which should be in the hands of the surveyor who uses this instrument. The limits of our space will allow only a brief reference to its principal features.

The main plate and standards resemble these parts of the compass.

a is the *latitude arc.*

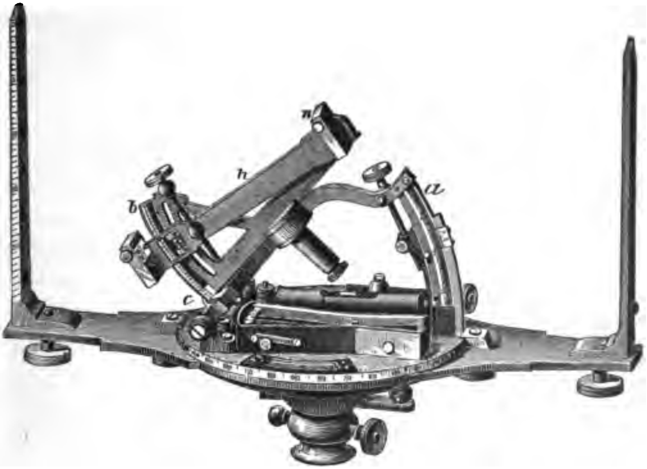
b is the *declination arc.*

h is an arm, on each end of which is a *solar lens* having its focus on a silvered plate on the other end.

c is the *hour arc.*

n is the *needle-box*, which has an arc of about 36°.

To run a north and south line with the solar compass. Set off the declination of the sun on the declination arc. Set off the latitude of the place (which may be determined by this instrument) on the latitude arc. Set the instrument over the station,



BURT'S SOLAR COMPASS.

level, and turn the sights in a north and south direction, approximately, by the needle. Turn the solar lens toward the sun, and bring the sun's image between the equatorial lines on the silvered plate. Allowance being made for refraction, the sights will then indicate a true north and south line.

The Solar Attachment for Transits.

This consists, essentially, of the solar apparatus of the solar compass attached to the telescope of the ordinary transit.

Laying Out the Public Lands.

The public lands north of the Ohio River and west of the Mississippi are generally laid out in townships approximately six miles square.

A **Principal Meridian,** or true north and south line, is first run by means of Burt's Solar Compass, and then an east and west line, called a **Base Line.**

Parallels to the base line are run at intervals of six miles, and north and south lines at the same intervals. Thus, the tract would be divided into townships exactly six miles square, if it were not for the convergence of the meridians on account of the curvature of the earth.

The north and south lines, or meridians, are called **Range Lines.** The east and west lines, or parallels, are called **Township Lines.**

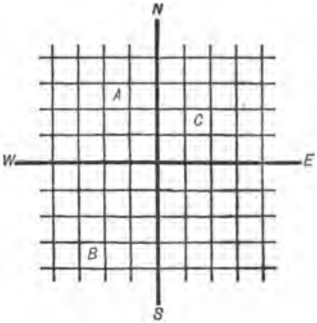


Fig. 33.

Let NS (Fig. 33) represent a principal meridian, and WE a base line; and let the other lines represent meridians and parallels at intervals of six miles.

The small squares, A, B, C, etc., will represent townships.

A would be designated thus: T. 3 N., R. 2 W.; that is, township three north, range two west; which means that the township is in the third tier north of the base line, and in the second tier west of the principal meridian. *B* and *C*, respectively, would be designated thus: T. 4 S., R. 3 W.; and T. 2 N., R. 2 E.

The townships are divided into sections approximately one mile square, and the sections are divided into quarter-sections. The township, section, and quarter-section corners are permanently marked.

6	5	4	3	2	1
7	8	9	10	11	12
18	17	16	15	14	13
19	20	21	22	23	24
30	29	28	27	26	25
31	32	33	34	35	36

Fig. 34.

The sections are numbered, beginning at the north-east corner, as in Fig. 34, which represents a township divided into sections. The quarter-sections are designated, according to their position, as N. E., N.W., S. E., and S.W.

Every fifth parallel is called a **Standard Parallel** or **Correction Line.**

Let *NS* (Fig. 35) represent a principal meridian; *WE* a base line; *rp*, etc., meridians; and *ms* the fifth parallel. If *Op* equals six miles, *mr* will be less than six miles on account of the convergence of the meridians. Surveyors are instructed to make *Op* such a distance that *mr* shall be six miles; then *mh*, *hk*, etc., are taken similarly. At the correction lines north of *ms* the same operation is repeated.

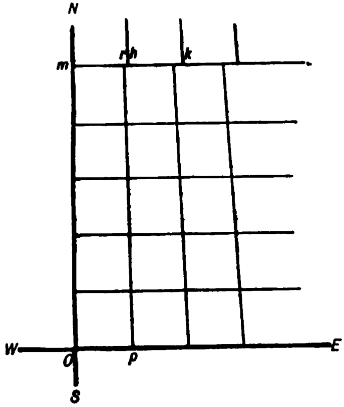


Fig. 35.

The township and section lines are surveyed in such an order as to throw the errors on the north and outer townships and sections.

If, in running a line, a navigable stream or a lake more than one mile in length is encountered, it is *meandered* by marking the intersection of the line with the bank and running lines from this point along the bank to prominent points which are marked, and the lengths and bearings of the connecting lines recorded.

Six principal meridians have been established and connected. In addition to these there are several independent meridians in the Western States and Territories which will in time be connected with each other and with the eastern system.

§ 26. PLANE–TABLE SURVEYING.*

After the principal lines of a survey have been determined and plotted, the details of the plot may be filled in by means of the plane-table; or, when a plot only of a tract of land is desired, this instrument affords the most expeditious means of obtaining it.

An approved form of the plane-table, as used in the United States Coast and Geodetic Survey, is shown in the plate on page 49.

The **Table-top** is a board of well-seasoned wood, panelled with the grain at right angles to prevent warping, and supported at a convenient height by a **Tripod** and **Levelling Head.**

The **Alidade** is a ruler of brass or steel supporting a telescope or sight standards, whose line of sight is parallel to a plane perpendicular to the lower side of the ruler, and embracing its fiducial edge.

The **Declinatoire** consists of a detached rectangular box containing a magnetic needle which moves over an arc of about 10° on each side of the 0.

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* In preparing this section the writer has consulted, by permission, the treatise on the plane-table by Mr. E. Hergesheimer, contained in the report for 1880 of the U.S. Coast and Geodetic Survey.

Two spirit levels at right angles are attached to the ruler or to the declinatoire. In some instruments these are replaced by a circular level, or by a detached spirit level.

The paper upon which the plot is to be made or completed is fastened evenly to the board by clamps, the surplus paper being loosely rolled under the sides of the board.

To place the table in position. This operation, which is sometimes called orienting the table, consists in placing the table so that the lines of the plot shall be parallel to the corresponding lines on the ground.

This may be accomplished by turning the table until the needle of the declinatoire indicates the same bearing as at a previous station, the edge of the declinatoire coinciding with the same line on the paper at both stations.

If, however, the line connecting the station at which the instrument is placed with another determined station is already plotted, the table may be placed in position by placing it over the station so that the plotted line is by estimation over and in the direction of the line on the ground; then making the edge of the ruler coincide with the plotted line, and turning the board until the line of sight bisects the signal at the other end of the line on the ground.

To plot any point. Let *ab* on the paper represent the line *AB* on the ground; it is required to plot *c*, representing *C* on the ground.

1. *By intersection.*

Place the table in position at *A* (Fig. 36), plumbing *a* over *A*, and making the fiducial edge of the ruler pass through *a*; turn the alidade about *a* until the line of sight bisects the signal at *C*, and draw a line along the fiducial edge of the ruler. Place the table in position at *B*, plumbing *b* over *B*, and repeat the operation just described. *c* will be the intersection of the two lines thus drawn.

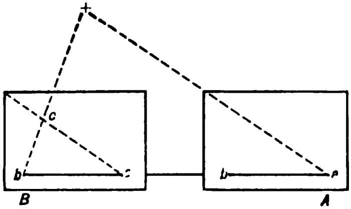
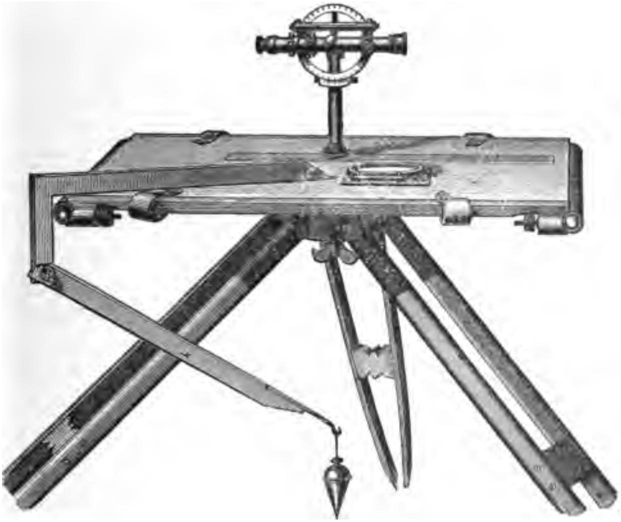


Fig. 36.



THE PLANE-TABLE.

2. *By resection.*

Place the table in position at A (Fig. 37), and draw a line in the direction of C, as in the former case; then remove the instrument to C, place it in position by the line drawn from a, make the edge of the ruler pass through b, and turn the alidade about b until B is in the line of sight. A line drawn along the edge of the ruler will intersect the line from a in c.

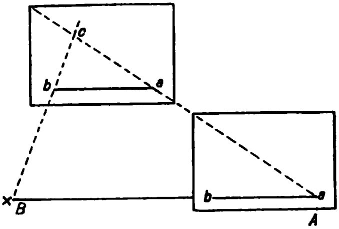


Fig. 37.

3. *By radiation.*

Place the table in position at A (Fig. 38), and draw a line from a toward C, as in the former cases. Measure AC, and lay off ac to the same scale as ab.

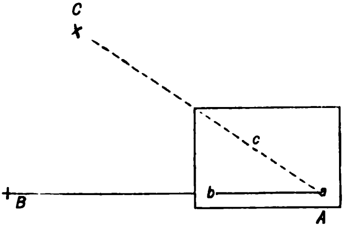


Fig. 38.

To plot a field $ABCD$.....

1. *By radiation.*

Set up the table at any point P, and mark p on the paper over P. Draw indefinite lines from p toward A, B, C, ... Measure PA, PB, ..., and lay off pa, pb, ..., to a suitable scale, and join a and b, b and c, c and d, etc.

2. *By progression.*

Set up the table at A, and draw a line from a toward B. Measure AB, and plot ab to a suitable scale. Set up the table in position at B, and in like manner determine and plot bc, etc.

3. *By intersection.*

Plot one side as a base line. Plot the other corners by the method of intersection, and join.

4. *By resection.*

Plot one side as a base line. Plot the other corners by the method of resection, and join.

The Three Point Problem.

Let *A*, *B*, *C* represent three points determined and plotted as *a*, *b*, *c* (Fig. 39); it is required to plot by the plane-table, *d* representing a fourth undetermined point *D*.

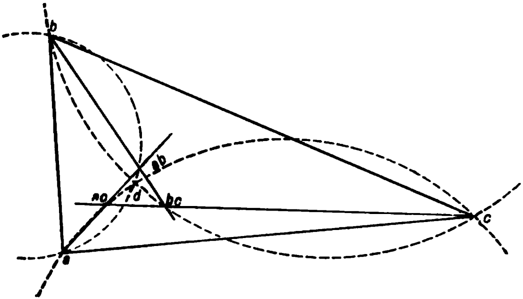


Fig. 39.

Determine the point *d* by the method of resection; first, from *A* and *B*, then from *A* and *C*. If the three lines from *a*, *b*, and *c* meet in a point, this will determine *d*. Ordinarily, they will not meet in a point, but will form a *triangle of error*, *ac bc ab*. Through *a*, *b*, and *ab*; *a*, *c*, and *ac*; and *b*, *c*, and *bc*, respectively, pass circumferences of circles; these three circles will intersect in the required point *d*.

CHAPTER III.

TRIANGULATION.*

§ 27. INTRODUCTORY REMARKS.

Geographical positions upon the surface of the earth are commonly determined by systems of triangles which connect a carefully determined base line with the points to be located.

Let F (Fig. 40) represent a point whose position with reference to the base line AB is required. Connect AB with F by the series of triangles ABC, ACD, ADE, and DEF, so that a signal at C is visible from A and B, a signal at D visible from A and C, a signal at E visible from A and D, and a signal at F visible from D and E. In the triangle ABC, the side AB is known, and the angles at A and B may be measured; hence, AC may be computed. In the triangle ACD, AC is known, and the angles at A and C may be measured; hence, AD may be computed. In like manner DE and EF or DF may be determined. DF, or some suitable line connected with DF, may be measured, and this result compared with the computed value to test the accuracy of the field measurements.

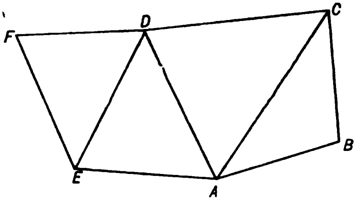


Fig. 40.

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* In preparing this chapter the writer has consulted, by permission, recent reports of the United States Coast and Geodetic Survey.

Three orders of triangulation are recognized; viz.:

Primary, in which the sides are from 20 to 150 miles in length.

Secondary, in which the sides are from 5 to 40 miles in length, and which connect the primary with the tertiary.

Tertiary, in which the sides are seldom over 5 miles in length, and which bring the survey down to such dimensions as to admit of the minor details being filled in by the compass and plane-table.

§ 28. THE MEASUREMENT OF BASE LINES.

Base lines should be measured with a degree of accuracy corresponding to their importance.

Suitable ground must be selected and cleared of all obstructions. Each extremity of the line may be marked by cross lines on the head of a copper tack driven into a stub which is sunk to the surface of the ground. Poles are set up in line about half a mile apart, the alignment being controlled by a transit placed over one end of the line.

The *preliminary measurement* may be made with an iron wire about one-eighth of an inch in diameter and 60^m in length. In measuring, the wire is brought into line by means of a transit set up in line not more than one-fourth of a mile in the rear. The end of each 60^m is marked with pencil lines on a wooden bench whose legs are thrust into the ground after its position has been approximately determined. If the last measurement exceeds or falls short of the extremity of the line, the difference may be measured with the 20^m chain.

The *final measurement* is made with the *base apparatus*, which consists of bars 6^m long, which are supported upon trestles when in use. These bars are placed end to end, and brought to a horizontal position, if this can be quickly accomplished; if not, the angle of inclination is taken by a sector, or a vertical offset is measured with the aid of a transit, so that the exact horizontal distance can be computed.

A thermometer is attached to each bar, so that the temperature of the bar may be noted and a correction for temperature applied.

The method of measuring lines varies according to the required degree of accuracy in any particular case, but the brief description given above will give the student a general idea of the methods employed.

§ 29. The Measurement of Angles.

Angles are measured by large theodolites, and the reading taken by microscopes to single seconds.

In order to eliminate errors of observation, and errors arising from imperfect graduation of the circles, a large number of readings is made and their mean taken. Two methods are in use; viz., repetition and series.

The method of *repetition* consists, essentially, in measuring the angles about a point singly, then taking two adjacent angles as a single angle, then three, etc.; thus "closing the horizon," or measuring the whole angular magnitude about a point in several different ways.

The method of *series* consists, essentially, in taking the readings of an angle with the circle or limb of the theodolite in one position, then turning the circle through an arc and taking the readings of the same angle again, etc.; thus reading the angle from successive portions of the graduated circle.

On account of the curvature of the earth, the sum of the three angles of a triangle upon its surface exceeds 180°. This *spherical excess*, as it is called, becomes appreciable only when the sides of the triangle are about 5 miles in length. To determine the angles of the rectilinear triangle having the same vertices, one-third of the spherical excess is deducted from each spherical angle.

CHAPTER IV.

LEVELLING.

§ 30. DEFINITIONS, CURVATURE, AND REFRACTION.

A **Level Surface** is a surface parallel with the surface of still water.

A **Level Line** is a line in a level surface.

Levelling is the process of finding the difference of level of two places, or the distance of one place above or below a level line through another place.

The **Line of Apparent Level** of a place is a tangent to the level line at that place. Hence, the line of apparent level is perpendicular to the plumb-line.

The **Correction for Curvature** is the deviation of the line of apparent level from the level line for any distance.

Let t (Fig. 41) represent the line of apparent level of the place P, a the level line, d the diameter of the earth; then c represents the correction for curvature. To compute the correction for curvature:

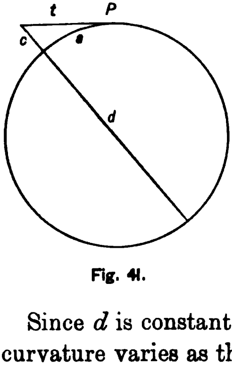


Fig. 41.

$$t^2 = c(c+d).$$

Therefore, $c = \dfrac{t^2}{c+d} = \dfrac{a^2}{d}$

approximately, since c is very small compared with d, and $t = a$ without appreciable error.

Since d is constant ($=$ 7920 miles, nearly), the correction for curvature varies as the square of the distance.

EXAMPLE. What is the correction for curvature for 1 mile?

By substituting in the formula deduced above, $c = \frac{a^2}{d} = \frac{1^2}{7920}$ mi. = 8 in., nearly.

Hence, the correction for curvature for any distance may be found in inches, approximately, by multiplying 8 by the square of the distance expressed in miles.

A correction for the refraction of the rays of light is sometimes made by decreasing the correction for curvature by one-sixth of itself.

§ 31. THE Y LEVEL.

This instrument is shown on page 59.

The *telescope* is about 20 inches in length, and rests on supports called *Y's*, from their shape. The *spirit level* is underneath the telescope, and attached to it. The *levelling-head* and *tripod* are similar to the same parts of the transit.

§ 32. THE LEVELLING ROD.

The rod shown in Fig. 42 is known as the New York levelling rod. It is made of two pieces of wood, sliding upon each other, and held together in any position by a clamp. The front surface of the rod is graduated to hundredths of a foot up to 6½ feet.

The target slides along the front of the rod, and is held in place by two springs which press upon the sides of the rod. It has a square opening at the centre, through which the division line of the rod opposite to the horizontal line of the target may be seen. It carries a vernier by which heights may be read to thousandths of a foot (§ 7).

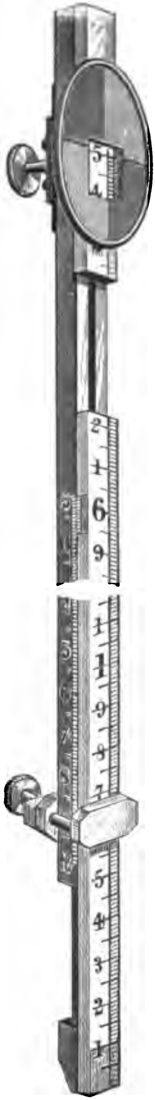


Fig. 42.

If a greater height than 6½ feet is desired, the target is made fast with its horizontal line at 6½ feet from the lower end. The back part of the rod is then moved up until the target is at the required height, and the reading is taken at the side of the rod. When fully extended the rod is 12 feet long.

§ 33. Difference of Level.

To find the difference of level between two places visible from an intermediate place.

Let *A* and *B* (Fig. 43) represent the two places. Set the Y level at a station equally distant, or nearly so, from *A* and *B*, but not necessarily on the line *AB*. Place the legs of the tripod firmly in the ground, and level over each opposite pair of levelling screws, successively. Let the rodman hold the levelling rod vertically at *A*. Bring the telescope to bear upon the rod (§ 8), and by signal direct the rodman to move the target until its horizontal line is in the line of apparent level of the telescope. Let the rodman now record the height *AA'* of the target. In like manner find *BB'*. The difference between *AA'* and *BB'* will be the difference of level required. If the instrument be equally distant from *A* and *B*, or nearly so, no correction for curvature will be necessary.

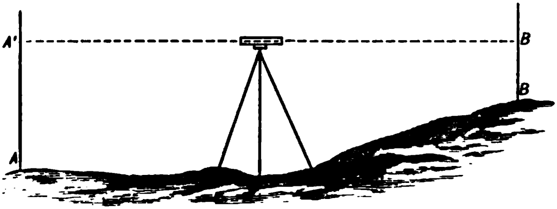
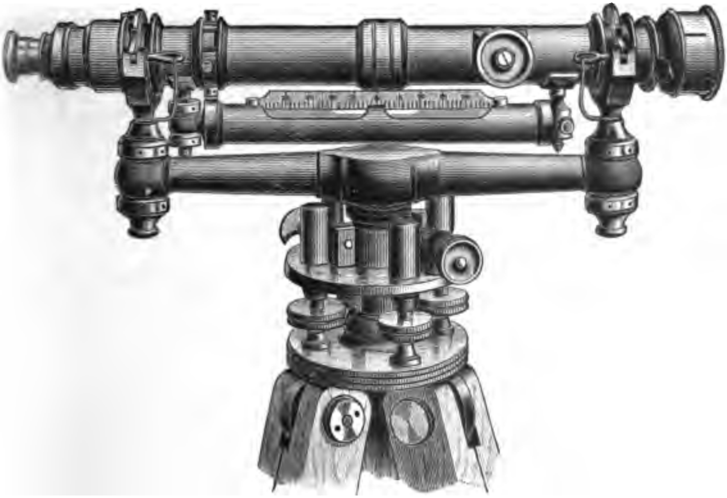


Fig. 43.

If the instrument be set up at one station, and the rod at the other, the difference between the heights of the optical axis of the telescope and the target, corrected for curvature and refraction, will be the difference of level required.



THE Y LEVEL.

To find the difference of level of two places, one of which cannot be seen from the other, and both invisible from the same place; or, when the two places differ considerably in level.

Let *A* and *D* (Fig. 44) represent the two places. Place the level midway between *A* and some intermediate station *B*.

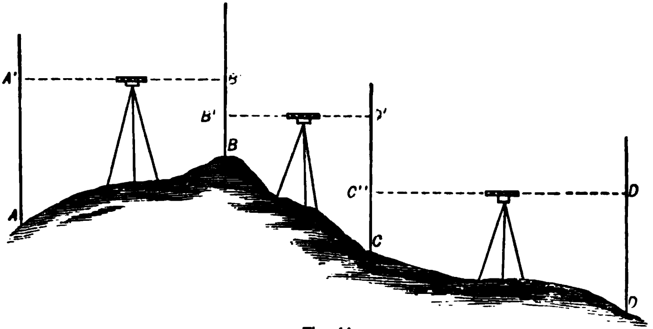


Fig. 44.

Find *AA'* and *BB'*, as in the preceding case, and record the former as a *back-sight* and the latter as a *fore-sight.* Select another intermediate station *C*, and in like manner find the back-sight *BB''* and the fore-sight *CC'*; and so continue until the place *D* is reached. The difference between the sum of the fore-sights and the sum of the back-sights will be the difference of level required.

§ 34. LEVELLING FOR SECTION.

The intersection of a vertical plane with the surface of the earth is called a **Section** or **Profile.** The term profile, however, usually designates the **Plot** or representation of the section on paper.

Levelling for Section is levelling to obtain the data necessary for making a profile or plot of any required section.

A profile is made for the purpose of exhibiting in a single view the inequalities of the surface of the ground for great distances along the line of some proposed improvement, such as a railroad, canal or ditch, and thus facilitating the establishment of the proper grades.

The data necessary for making a profile of any required section are, the heights of its different points above some assumed horizontal plane, called the **Datum Plane,** together with their horizontal distances apart or from the beginning of the section.

The position of the datum plane is fixed with reference to some permanent object near the beginning of the section, called a **Bench Mark,** and, in order to avoid negative heights, is assumed at such a distance below this mark that all the points of the section shall be above it.

The heights of the different points of the section above the datum plane are determined by means of the level and levelling-rod; and the horizontal length of the section is measured with an engineer's chain or tape, and divided into equal parts, one hundred feet in length, called **Stations,** marked by stakes numbered 0, 1, 2, 3, etc.

Where the ground is very irregular, it may be necessary, besides taking sights at the regular stakes, to take occasional sights at points between them. If, for instance, at a point sixty feet in advance of stake 8 there is a sudden rise or fall in the surface, the height of this point would be determined and recorded as at stake 8.60.

The readings of the rod are ordinarily taken to the nearest tenth of a foot, except on *bench marks* and points called *turning points*, where they are taken to thousandths of a foot.

A Turning Point is a point on which the last sight is taken just before changing the position of the level, and the first sight from the new position of the instrument. A turning point may be coincident with one of the stakes, but must always be a hard point, so that the foot of the rod may stand at the same level for both readings.

To explain the method of obtaining the field notes necessary for making a profile, let 0, 1, 2, 3 ..... 11 (Fig. 45) represent a portion of a section to be levelled and plotted. Establish a bench mark at or near the beginning of the line, measure the horizontal length of the section, and set stakes one hundred feet apart, numbering them 0, 1, 2, 3, etc. Place the level at some point, as between 2 and 3, and take the reading of the rod on the bench = 4.832. Let PP' represent the datum plane, say 15 feet below the bench mark, then

$$15 + 4.832 = 19.832$$

will be the height of the line of sight AB, called the **Height of the Instrument**, above the datum plane. Now take the reading at $0 = 5.2 = 0A$, and subtract the same from 19.832, which leaves $14.6 = 0P$, the height of the point 0 above the datum plane. Next take sights at 1, 2, 3, 3.40, and 4 equal respectively to 3.7, 3.0, 5.1, 4.8, and 8.3, and subtract the same from 19.832; the remainders 16.1, 16.8, 14.7, 15.0, and 11.5 will be the respective heights of the points 1, 2, 3, 3.40, and 4. Then, as it will be necessary to change the position of the instrument, select a point in the neighborhood of 4 suitable as a turning point (*t.p.* in the figure), and take a careful reading on it = 8.480; subtract this from 19.832, and the remainder, 11.352, will be the height of the turning point. Now carry the instrument forward to a new position, as between 5 and 6, shown in the figure, while the rodman remains at *t.p.*; take a second reading

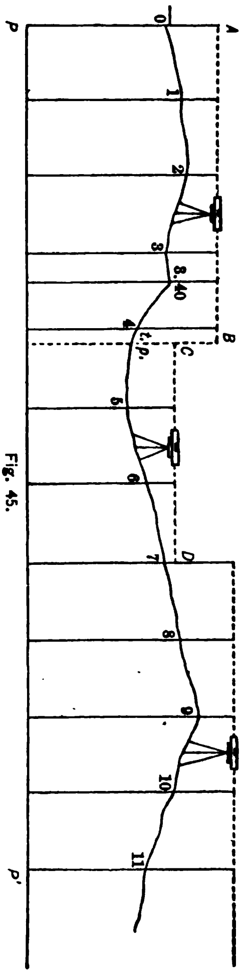


Fig. 45.

on *t.p.* = 4.102, and add it to 11.352, the height of *t.p.* above *PP'*; the sum 15.454 will be the height of the instrument *CD* in its new position. Now take sight upon 5, 6, and 7, equal respectively to 4.9, 2.8, and 0.904; subtract these sights from 15.454, and the results 10.6, 12.7, and 14.550 will be the heights of the points 5, 6, and 7 respectively. The point 7, being suitable, is made a turning point, and the instrument is moved forward to a point between 9 and 10. The sight at 7 = 6.870 added to the height of 7 gives 21.420 as the height of the instrument *EF* in its new position. The readings at 8, 9, 10, and 11, which are respectively 5.4, 3.6, 5.8, and 9.0, subtracted from 21.420, will give the heights of these points, namely, 16.0, 17.8, 15.6, and 12.4.

Proceed in like manner until the entire section is levelled, establishing bench marks at intervals along the line to serve as reference points for future operations. A record of the work described above is kept in a field book as follows:

Station.	+ *S.*	*H.I.*	− *S.*	*H.S.*	*Remarks.*
B	4.832			15.	Bench on rock 20 ft.
0		19.832	5.2	14.6	south of 0.
1			3.7	16.1	
2			3.0	16.8	
3			5.1	14.7	3 to 3.40 turnpike road.
3.40			4.8	15.0	
4			8.3	11.5	
t.p.	4.102		8.480	11.352	
5		15.454	4.9	10.6	
6			28.	12.7	
7	6.870		0.904	14.550	
8		21.420	5.4	16.0	
9			3.6	17.8	
10			5.8	15.6	
11			9.0	12.4	
B					Bench on oak stump
12					27 ft. N.E. of 12,
etc.					etc.

The first column contains the numbers or names of all the points on which sights are taken. The second column contains the sight taken on the first bench mark, and the sight on each turning point taken immediately after the instrument has been moved to a new position. These are called **Plus Sights** (+ *S.*) because they are **added** to the heights of the points on which they are taken to obtain the height of the instrument given in the third column (*H.I.*). The fourth column contains all the readings except those recorded in the second column. These are called **Minus Sights** (— *S.*) because they are **subtracted** from the numbers in the third column to obtain all the numbers in the fifth column except the first, which is the assumed depth of the datum plane below the bench. The fifth column contains the required heights of all the points of the section named in the first column together with the heights of all benches and turning points.

To find the difference of level between any two points of the section, we have only to take the difference between the numbers in the fifth column opposite these points.

The real field notes are contained in the first, second, fourth, and last columns; the other columns may be filled after the field operations are completed. The field book may contain other columns; one for **height of grade** (*H.G.*), another for **depth of cut** (*C.*), and another for **height of embankment** or **fill** (*F.*).

To plot the section. Draw a line *PP′* (Fig. 45), to represent the datum plane, and beginning at some point as *P*, lay off the distances 100, 200, 300, 340, 400 feet, etc., to the right, using some convenient scale, say 250 feet to the inch, as in the figure. At these points of division erect perpendiculars equal in length to the height of the points 0, 1, 2, 3.40, 4, etc., given in the fifth column of the above field notes, using in this case a larger scale, say 20 feet to the inch. Through the extremities of these perpendiculars draw the irregular line 0, 1, 2, 3 ..... 11, and the result, with some explanatory figures, will be the required plot or profile.

The making of a profile is much simplified by the use of *profile paper*, which may be had by the yard or roll.

If a *horizontal plot* is required, the bearings of the different portions of the section must be taken.

§ 35. SUBSTITUTES FOR THE Y LEVEL.

For many purposes not requiring accuracy, the following simple instruments in connection with a graduated rod will be found sufficient.

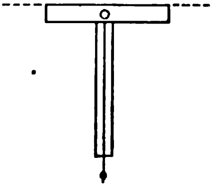


Fig. 46.

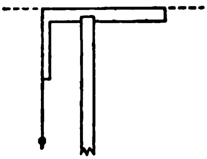


Fig. 47.

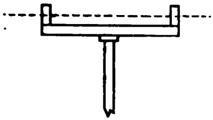


Fig. 48.

The **Plumb Level** (Fig. 46) consists of two pieces of wood joined at right angles. A straight line is drawn on the upright perpendicular to the upper edge of the cross-head.

The instrument is fastened to a support by a screw through the centre of the cross-head. The upper edge of the cross-head is brought to a level by making the line on the upright coincide with a plumb-line.

A modified form is shown in Fig. 47. A carpenter's square is supported by a post, the top of which is split or sawed so as to receive the longer arm. The shorter arm is made vertical by a plumb-line which brings the longer arm to a level.

The **Water Level** is shown in Fig. 48. The upright tubes are of glass, cemented into a connecting tube of any suitable material. The whole is nearly filled with water, and supported at a convenient height. The surface of the water in the uprights determines the level.

By sighting along the upper surface of the block in which the **Spirit Level** is mounted for the use of mechanics, a level line may be obtained.

EXERCISE V.

1. Find the difference of level of two places from the following field notes: back-sights, 5.2, 6.8, and 4.0; fore-sights, 8.1, 9.5, and 7.9.

2. Write the proper numbers in the third and fifth columns of the following table of field notes, and make a profile of the section.

Station.	+ *S.*	*H.I.*	− *S.*	*H.S.*	*Remarks.*
B	6.944			20.	Bench on post 22 ft. north of 0.
0			7.4		
1			5.6		
2			3.9		
3			4.6		
t.p.	3.855		5.513		
4			4.9		
5			3.5		
6			1.2		

3. Stake 0 of the following notes stands at the lowest point of a pond to be drained into a creek; stake 10 stands at the edge of the bank, and 10.25 at the bottom of the creek. Make a profile, draw the grade line through 0 and 10.25, and fill out the columns *H.G.* and *C.*, the former to show the height of grade line above the datum, and the latter, the depths of cut at the several stakes necessary to construct the drain.

Station.	+ *S.*	*H.I.*	− *S.*	*H.S.*	*H.G.*	*C.*	*Remarks.*
B	6.000			25.			Bench on rock 30 feet west of stake 1.
0			10.2		20.8	0.0	
1			5.3			5.3	
2			4.6				
3			4.0				
4			6.8				
5	4.572		7.090				
6			3.9				
7			2.0				
8			4.9				
9			4.3				
10.			4.5				
10.25			11.8				

§ 36. TOPOGRAPHICAL LEVELLING.

The principal object of topographical surveying is to show the contour of the ground. This operation, called topographical levelling, is performed by representing on paper the curved lines in which parallel horizontal planes at uniform distances apart would meet the surface.

It is evident that all points in the intersection of a horizontal plane with the surface of the ground are at the same level. Hence, it is only necessary to find points at the same level, and join these to determine a line of intersection.

The method commonly employed will be understood by a reference to Fig. 49. The ground *ABCD* is divided into equal squares, and a numbered stake driven at each intersection. By means of a level and levelling rod the heights of the other stations above *m* and *D*, the lowest stations, are determined. A plot of the ground with the intersecting lines is then drawn, and the height of each station written as in the figure.

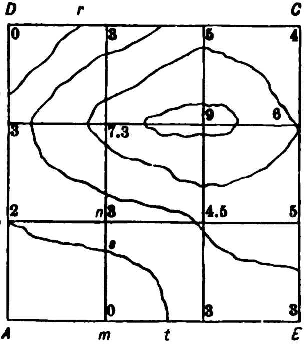


Fig 49.

Suppose that the horizontal planes are 2 feet apart; if the first passes through *m* and *D*, the second will pass through *p*, which is 2 feet above *m*; and since *n* is 3 feet above *m*, the second plane will cut the line *mn* in a point *s* determined by the proportion $mn : ms :: 3 : 2$. In like manner the points *t*, *q*, and *r* are determined.

The irregular line *tsp* ..... *qr* represents the intersection of the second horizontal plane with the surface of the ground. In like manner the intersections of the planes, respectively, 4, 6, and 8 feet above *m*, are traced.

It is evident that the more rapid the change in level the nearer these lines will approach each other.

CHAPTER V.

RAILROAD SURVEYING.

§ 37. General Remarks.

When the general route of a railroad has been determined, a middle surface line is run with the transit. A profile of this line is determined, as in § 34. The levelling stations are commonly 1 chain (100 feet) apart. Places of different level are connected by a gradient line, which intersects the perpendiculars to the datum line at the levelling stations in points determined by simple proportion. Hence, the distance of each levelling station, above or below the level or gradient line which represents the position of the road bed, is known.

§ 38. Cross Section Work.

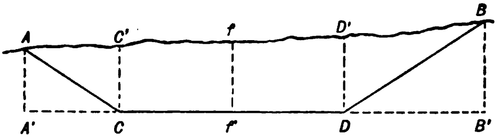


Fig. 50.

Excavations. If the road bed lies below the surface, an excavation is made.

Let *ACDB* (Fig. 50) represent a cross section of an excavation, *f* a point in the middle surface line, *f'* the corresponding point in the road bed, and *CD* the width of the excavation at the bottom. The slopes at the sides are commonly made so that $AA' = \frac{2}{3}A'C$, and $BB' = \frac{2}{3}DB'$. ff' and CD being known, the points A, B, C', and D' are readily determined by a level and tape measure.

If from the area of the trapezoid $ABB'A'$ the areas of the triangles $AA'C$ and $BB'D$ be deducted, the remainder will be the area of the cross section.

In like manner the cross section at the next station may be determined. These two cross sections will be the bases of a frustum of a quadrangular pyramid whose volume will be the amount of the excavation, approximately.

Embankments. If the road bed lies above the surface, an embankment is made, the cross section of which is like that of the excavation, but inverted.

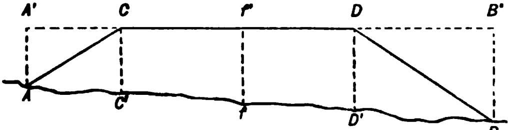


Fig. 51.

Fig. 51 represents the cross section of an embankment which is lettered so as to show its relation to Fig. 50.

§ 39. Railroad Curves.

When it is necessary to change the direction of a railroad, it is done gradually by a curve, usually the arc of a circle.

Let AB and AO (Fig. 52) represent two lines to be thus connected. Take $AD = AE = t$. The intersection of the perpendiculars DC and EC determines the centre C, and the radius of curvature $DC = r$. The length of the radius

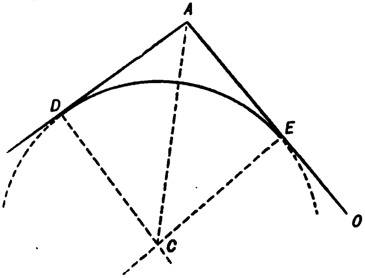


Fig. 52.

of curvature will depend on the angle A and the tangent AD. For, in the right triangle ADC,

$$\tan DAC = \frac{DC}{AD}, \text{ or } \tan \tfrac{1}{2} A = \frac{r}{t}$$

Hence, $r = t \tan \tfrac{1}{2} A$.

To Lay out the Curve.

First Method. Let Dm (Fig. 53) represent a portion of the tangent. It is required to find mP, the perpendicular to the tangent meeting the curve at P.

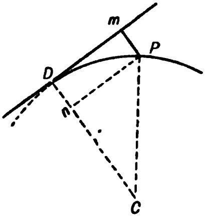


Fig. 53.

$$mP = Dn = CD - Cn.$$

But $CD = r$,

and $Cn = \sqrt{\overline{CP}^2 - \overline{Pn}^2}$

$= \sqrt{r^2 - t^2}$.

Hence, $mP = r - \sqrt{r^2 - t^2}$.

Second Method. It is required to find mP (Fig. 54) in the direction of the centre.

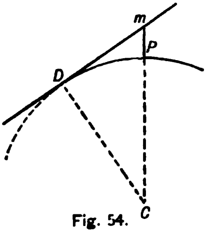


Fig. 54.

$$mP = mC - PC.$$

But $mC = \sqrt{\overline{DC}^2 + \overline{Dm}^2} = \sqrt{r^2 + t^2}$.

Hence,

$$mP = \sqrt{r^2 + t^2} - r.$$

Third Method. Place transits at D and E (Fig. 55). Direct the telescope of the former to E, and of the latter to A. Turn each toward the curve the same number of degrees, and mark P, the point of intersection of the lines of sight. P will be a point in the circle to which AD and AE are tangents at D and E, respectively.

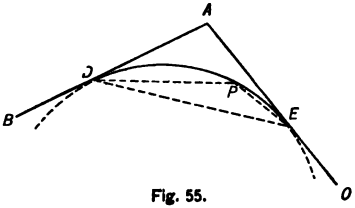


Fig. 55.

Fourth Method. Measure the angle A and the tangents $AD = AE$ (Fig. 56). Compute r. Select any distance l (the length of the chain will be found convenient). Place the transit at D and direct toward A. Turn off an angle ADP determined by the equation $\sin ADP = \frac{l}{2r}$, and measure $DP = l$; make the angle $ADP' = 2ADP$ and $PP' = l$; make the angle $ADP'' = 3ADP$ and $P'P'' = l$; etc. P, P', P'', etc., will be in the circumference of the required arc.

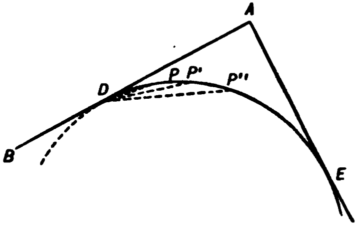


Fig. 56.

ANSWERS.

PLANE TRIGONOMETRY.

EXERCISE I.

1. $\sin B = \frac{b}{c}$, $\cos B = \frac{a}{c}$, $\tan B = \frac{b}{a}$, $\cot B = \frac{a}{b}$, $\sec B = \frac{c}{a}$, $\csc B = \frac{c}{b}$

3. (i.) $\sin = \frac{3}{5}$, $\cos = \frac{4}{5}$, (ii.) $\sin = \frac{5}{13}$, etc. (v.) $\sin = \frac{20}{29}$, etc.
$\tan = \frac{3}{4}$, $\cot = \frac{4}{3}$, (iii.) $\sin = \frac{8}{17}$, etc. (vi.) $\sin = \frac{119}{169}$, etc.
$\sec = \frac{5}{4}$, $\csc = \frac{5}{3}$. (iv.) $\sin = \frac{9}{41}$, etc.

4. The required condition is that $a^2 + b^2 = c^2$. It is.

5. (i.) $\sin = \frac{2mn}{m^2+n^2}$, etc. (iii.) $\sin = \frac{q}{s}$, etc.
(ii.) $\sin = \frac{2xy}{x^2+y^2}$, etc. (iv.) $\sin = \frac{ms}{qr}$, etc.

7. In (iii.) $p^2q^2 + q^2s^2 = p^2s^2$; in (iv.) $m^2n^2s^2 + m^2p^2v^2 = n^2q^2r^2$.

8. $c = 145$; whence, $\sin A = \frac{24}{145} = \cos B$; $\cos A = \frac{143}{145} = \sin B$;
$\tan A = \frac{24}{143} = \cot B$; $\cot A = \frac{143}{24} = \tan B$; $\sec A = \frac{145}{143} = \csc B$; etc.

9. $b = 0.023$; whence, $\tan A = \cot B = \frac{23}{264}$; $\cot A = \tan B = \frac{264}{23}$, etc.

10. $a = 16.8$; whence, $\sin A = \frac{168}{193} = \cos B$, etc.

11. $c = p + q$; whence, $\sin A = \frac{\sqrt{p^2+q^2}}{p+q} = \cos B$; etc.

12. $b = \sqrt{q(p+q)}$; whence, $\tan A = \sqrt{\frac{p}{q}} = \cot B$; etc.

13. $a = p - q$; whence, $\sin A = \frac{p-q}{p+q} = \cos B$; etc.

14 $\sin A = \frac{2}{5}\sqrt{5} = 0.89443$; etc. 15. $\sin A = \frac{2}{3}$; etc.

16. $\sin A = \frac{1}{8}(5 + \sqrt{7}) = 0.95572$; etc.

17. $\cos A = \frac{1}{8}(\sqrt{31} - 1) = 0.57097$; $\sin A = \frac{1}{8}(\sqrt{31} + 1) = 0.82097$; etc.

18 $x = 12.3$. 20. $a = 9$. 22. $c = 40$.

19. $b = 1.54$. 21. $b = 68$. 23. $c = 229.62$.

24. Construct a rt. △ with legs equal to 3 and 2 respectively; then construct a similar △ with hypotenuse equal to 6.

In like manner, 25, 26, 27, may be solved.

28. $a = 1.5$ miles; $b = 2$ miles. 31. 400,000 miles.

30. $a = 0.342$, $b = 0.940$; $a = 1.368$, $b = 3.760$. 32. 142.926 yards.

EXERCISE II.

5. Through A (Fig. 3) draw a tangent, and take $AT = 3$; the angle AOT is the required angle.

6. From O (Fig. 3) as a centre, with a radius $= 2$, describe an arc cutting at S the tangent drawn through B; the angle SOA is the required angle.

7. In Fig. 3, take $OM = \frac{1}{2}$, and erect $MP \perp OA$ and intersecting the circumference at P; the angle POM is the required angle.

8. Since $\sin x = \cos x$, $OM = PM$ (Fig. 3), and $x = 45°$; hence, construct $x = 45°$.

9. Construct a rt. △ with one leg = twice the other; the angle opposite the longer leg is the required angle.

10. Divide OA (Fig. 3) into four equal parts; at the first point of division from O erect a perpendicular to meet the circumference at some point P. Join OP; the angle AOP is the required angle.

21. $r \sin x$. 22. Leg adjacent to $A = nc$, leg opposite to $A = mc$.

EXERCISE III.

1. $\cos 60°$. $\cot 1°$. $\sec 71° 50'$. $\tan 7° 41'$.
$\sin 45°$. $\tan 75°$. $\sin 52° 36'$. $\sec 35° 14'$.

2. $\cos 30°$. $\cot 33°$. $\sec 20° 58'$. $\tan 0° 1'$.
$\sin 15°$. $\tan 6°$. $\sin 4° 21'$. $\sec 44° 59$.

3. $\frac{1}{3}\sqrt{3}$.

4. $\tan A = \cot A = \cot(90° - A)$; hence, $A = 90° - A$ and $A = 45°$.

5. 30°. 7. 90°. 9. 22° 30′. 11. 10°.

6. 30° 8. 60°. 10. 18°. 12. $\frac{90°}{n+1}$.

EXERCISE V.

1. $\cos A = \frac{5}{13}$, $\tan A = \frac{12}{5}$, $\cot A = \frac{5}{12}$, $\sec A = \frac{13}{5}$, $\csc A = \frac{13}{12}$.
2. $\cos A = 0.6$, $\tan A = 1.3333$, $\cot A = 0.75$, $\sec A = 1.6667$, $\csc A = 1.25$.
3. $\sin A = \frac{11}{61}$, $\tan A = \frac{11}{60}$, $\cot A = \frac{60}{11}$, $\sec A = \frac{61}{60}$, $\csc A = \frac{61}{11}$.
4. $\sin A = 0.96$, $\tan A = 3.42854$, $\cot A = 0.29167$, $\sec A = 3.5714$.
5. $\sin A = 0.8$, $\cos A = 0.6$, $\cot A = 0.75$, $\sec A = 1.6667$, $\csc A = 1.25$.
6. $\sin A = \frac{1}{2}\sqrt{2}$, $\cos A = \frac{1}{2}\sqrt{2}$, $\tan A = 1$, $\sec A = \sqrt{2}$, $\csc A = \sqrt{2}$.
7. $\tan A = 2$, $\sin A = 0.90$, $\cos A = 0.45$, $\sec A = 2.22$, $\csc A = 1.11$.

8. $\cos A=\frac{1}{2}$, $\sin A=\frac{1}{2}\sqrt{3}$, $\tan A=\sqrt{3}$, $\cot A=\frac{1}{3}\sqrt{3}$, $\csc A=\frac{2}{3}\sqrt{3}$.

9. $\sin A=\frac{1}{2}\sqrt{2}$, $\cos A=\frac{1}{2}\sqrt{2}$, $\tan A=1$, $\cot A=1$, $\sec A=\sqrt{2}$.

10. $\cos A=\sqrt{1-m^2}$, $\tan A=\frac{m}{1-m^2}\sqrt{1-m^2}$, $\cot A=\frac{1}{m}\sqrt{1-m^2}$.

11. $\cos A=\frac{1-m^2}{1+m^2}$, $\tan A=\frac{2m}{1-m^2}$, $\cot A=\frac{1-m^2}{2m}$, $\sec A=\frac{1+m^2}{1-m^2}$.

12. $\sin A=\frac{m^2-n^2}{m^2+n^2}$, $\tan A=\frac{m^2-n^2}{2mn}$, $\sec A=\frac{m^2+n^2}{2mn}$.

13. $\cot=1$, $\sin=\frac{1}{2}\sqrt{2}$, $\cos=\frac{1}{2}\sqrt{2}$, $\sec=\sqrt{2}$, $\csc=\sqrt{2}$.

14. $\cos=\frac{1}{2}\sqrt{3}$, $\tan=\frac{1}{3}\sqrt{3}$, $\cot=\sqrt{3}$, $\sec=\frac{2}{3}\sqrt{3}$, $\csc=2$.

15. $\sin=\frac{1}{2}\sqrt{3}$, $\cos=\frac{1}{2}$, $\tan=\sqrt{3}$, $\cot=\frac{1}{3}\sqrt{3}$, $\sec=2$.

16. $\sin=\frac{1}{2}\sqrt{2-\sqrt{3}}$, $\cos=\frac{1}{2}\sqrt{2+\sqrt{3}}$, $\cot=2+\sqrt{3}$.

17. $\sin=\frac{1}{2}\sqrt{2-\sqrt{2}}$, $\cos=\frac{1}{2}\sqrt{2+\sqrt{2}}$, $\tan=\sqrt{2}-1$.

18. $\cos=1$, $\tan=0$, $\cot=\infty$, $\sec=1$, $\csc=\infty$.

19. $\cos=0$, $\tan=\infty$, $\cot=0$, $\sec=\infty$, $\csc=1$.

20. $\sin=1$, $\cos=0$, $\cot=0$, $\sec=\infty$, $\csc=1$.

21. $\cos A=\sqrt{1-\sin^2 A}$, $\tan A=\frac{\sin A}{\sqrt{1-\sin^2 A}}$, $\csc A=\frac{1}{\sin A}$.

22. $\sin A=\sqrt{1-\cos^2 A}$, $\tan A=\frac{\sqrt{1-\cos^2 A}}{\cos A}$, $\cot A=\frac{\cos A}{\sqrt{1-\cos^2 A}}$,
$\sec A=\frac{1}{\cos A}$, $\csc A=\frac{1}{\sqrt{1-\cos^2 A}}$.

23. $\sin A=\frac{\tan A}{\sqrt{1+\tan^2 A}}$, $\cos A=\frac{1}{\sqrt{1+\tan^2 A}}$, $\cot A=\frac{1}{\tan A}$,
$\sec A=\sqrt{1+\tan^2 A}$, $\csc A=\frac{\sqrt{1+\tan^2 A}}{\tan A}$.

24. $\tan A=\frac{1}{\cot A}$, $\csc A=\sqrt{1+\cot^2 A}$, $\sin A=\frac{1}{\sqrt{1+\cot^2 A}}$,
$\cos A=\frac{\cot A}{\sqrt{1+\cot^2 A}}$, $\sec A=\frac{\sqrt{1+\cot^2 A}}{\cot A}$.

25. $\sin A=\frac{1}{5}\sqrt{5}$, $\cos A=\frac{2}{5}\sqrt{5}$.

26. $\sin A=\frac{1}{4}\sqrt{15}$, $\tan A=\sqrt{15}$.

27. $\sin A=\frac{9}{41}$, $\cos A=\frac{40}{41}$.

28. $\frac{1-3\cos^2 A+3\cos^4 A}{\cos^2 A-\cos^4 A}$.

EXERCISE VI.

1. $\frac{b}{c} = \cos A$; $\therefore c = \frac{b}{\cos A}$.

5. $A = 90° - B$, $a = c \cos B$, $b = c \sin B$.

39. $c = 7.8112$, $A = 39° 48'$, $B = 50° 12'$.

40. $b = 69.99$, $A = 30' 12''$, $B = 89° 29' 48''$.

41. $a = 1.1886$, $A = 43° 20'$, $B = 46° 40'$.

42. $b = 21.249$, $c = 22.372$, $B = 71° 46'$.

43. $a = 6.6882$, $c = 13.738$, $B = 60° 52'$.

44. $a = 63.89$, $b = 23.369$, $B = 20° \ 6'$.

45. $a = 19.40$, $b = 18.778$, $A = 45° 56'$.

46. $b = 53.719$, $c = 71.377$, $A = 41° 11'$.

47. $a = 12.981$, $c = 15.796$, $A = 55° 16'$.

48. $a = 0.58046$, $b = 8.442$, $A = 3° 56'$.

49. $F = \frac{1}{2}(c^2 \sin A \cos A)$.

50. $F = \frac{1}{2}(a^2 \cot A)$.

51. $F = \frac{1}{2}(b^2 \tan A)$.

52. $F = \frac{1}{2}(a\sqrt{c^2 - a^2})$.

53. $b = 11.6$, $c = 15.315$, $A = 40° 45' 48''$, $B = 49° 14' 12''$.

54. $a = 7.2$, $c = 8.766$, $B = 34° 46' 40''$, $A = 55° 13' 20''$.

55. $a = 3.6474$, $b = 6.58$, $c = 7.5233$, $B = 61°$.

56. $a = 10.283$, $b = 19.449$. $A = 27° 52'$, $B = 62° 8'$.

57. 19° 28′ and 70° 32′.

58. 3 and 5.1961.

59. $a = c \cos \frac{90°}{n+1}$,
$b = c \sin \frac{90°}{n+1}$.

60. 36° 52′ 12″ and 53° 7′ 48″.

61. 212.1 feet.

62. 732.22 feet.

63. 3270 feet.

64. 37.3 feet, 96 feet.

65. $\tan A = \frac{a}{b}$, $A = 59° 45'$.

66. $a = b \tan A$, 95.34.

67. 1° 25′ 56″.

68. 7.0712 miles in each direction

69. 20.88 feet.

70. 56.65 feet.

71. 228.63 yards.

72. 136.6 feet.

73. 140 feet.

74. 84.74 feet.

EXERCISE VII.

1. $C = 2(90° - A)$, $c = 2a \cos A$, $h = a \sin A$.
2. $A = \frac{1}{2}(180° - C)$, $c = 2a \cos A$, $h = a \sin A$.
3. $C = 2(90° - A)$, $a = c \div 2 \cos A$, $h = a \sin A$.

4. $A = \frac{1}{2}(180° - C)$, $a = c + 2\cos A$, $h = a \sin A$.
5. $C = 2\ (90° - A)$, $a = h + \sin A$, $c = 2a\cos A$.
6. $A = \frac{1}{2}(180° - C)$, $a = h + \sin A$, $c = 2a\cos A$.
7. $\sin A = h \div a$, $C = 2(90° - A)$, $c = 2a\cos A$.
8. $\tan A = h \div \frac{1}{2}c$, $C = 2(90° - A)$, $a = h + \sin A$.
9. $A = 67° 22' 50''$, $C = 45° 14' 20''$, $h = 13.2$.
10. $c = 0.21943$, $h = 0.27384$, $F = 0.03004$.
11. $a = 2.055$, $h = 1.6852$, $F = 1.9819$.
12. $a = 7.705$, $c = 3.6676$, $F = 13.73$.
13. $A = 79° 36' 30''$, $C = 20° 47'$, $c = 2.4206$.
14. $A = 77° 19' 11''$, $C = 25° 21' 38''$, $a = 20.5$.
15. $A = 25° 28'$, $C = 129° 4'$, $a = 81.388$, $h = 35$.
16. $A = 81° 12' 9''$, $C = 17° 35' 42''$, $a = 17$, $c = 5.2$.

17. $F = \frac{1}{4}c\sqrt{4a^2 - c^2}$.
18. $F = a^2 \sin\frac{1}{2}C \cos\frac{1}{2}C$.
19. $F = a^2 \sin A \cos A$.
20. $F = h^2 \tan\frac{1}{2}C$.
21. 28.284 feet, 4525.44 sq. feet.
22. 0.76537.
23. 94° 20′.
24. 2.7261.
25. 38° 56′ 33″.
26. 37.7

EXERCISE VIII.

1. $r = 1.618$, $h = 1.5388$, $F = 7.694$.
2. $r = 11.269$, $h = 10.886$, $F = 381.04$.
3. $h = 0.9848$, $p = 6.2514$, $F = 3.0782$.
4. $h = 19.754$, $c = 6.2537$, $F = 1236$.
5. $r = 1.0824$, $c = 0.8284$, $F = 3.3137$.
6. $r = 2.592$, $h = 2.488$, $c = 1.4615$.
7. $r = 1.5994$, $h = 1.441$, $p = 9.716$.

8. 0.6181.
9. 0.64984.
10. 0.51764.
11. $b = \dfrac{c}{2\cos\dfrac{90°}{n}}$.
12. 0.2238.
13. 0.31.
14. 0.82842.
15. 94.63.
16. 415.
17. 11.636.
18. 99.64.
19. 1.0235.
20. 0.635.

EXERCISE IX.

5. Two angles: one in Quadrant I., the other in Quadrant II.

6. Four values: two in Quadrant I., two in Quadrant IV.

7. x may have two values in the first case, and one value in each of the other cases.

8. If $\cos x = -\frac{2}{9}$, x is between 90° and 270°; if $\cot x = 4$, x is between 0° and 90° or 180° and 270°; if $\sec x = 80$, x is between 0° and 90° or between 270° and 360°; if $\csc x = -3$, x is between 180° and 360°.

9. In Quadrant III.; in Quadrant II.; in Quadrant III.

10. 40 angles; 20 positive and 20 negative:

11. +, when x is known to be in Quadrant I. or IV.; −, when x is known to be in Quadrant II. or III.

14. $\sin x = -\frac{4}{7}\sqrt{3}$, $\tan x = -4\sqrt{3}$, $\cot x = -\frac{1}{12}\sqrt{3}$, $\csc x = -\frac{7}{12}\sqrt{3}$.

15. $\sin x = \pm\frac{1}{10}\sqrt{10}$, $\cos x = \mp\frac{3}{10}\sqrt{10}$, $\tan x = -\frac{1}{3}$, $\sec x = \mp\frac{1}{3}\sqrt{10}$.
$\csc x = \pm\sqrt{10}$.

16. The cosine, the tangent, the cotangent, and the secant are negative when the angle is obtuse.

17. Sine and cosecant leave it doubtful whether the angle is an acute angle or an obtuse angle; the other functions, if + determine an acute angle, if − an obtuse angle.

20. $\sin 450 = \sin(360 + 90) = \sin 90 = 1$; $\tan 540° = \tan 180° = 0$;
$\cos 630° = \cos 270° = 0$; $\cot 720° = \cot 0° = \infty$;
$\sin 810° = \sin 90° = 1$; $\csc 900° = \csc 180° = \infty$.

21. 45°, 135°, 225°, 315°. 22. 0. 23. 0. 24. 0.

25. $a^2 - b^2 + 4ab$.

EXERCISE X.

2. $\sin 172° = \sin 8°$.
3. $\cos 100° = -\sin 10°$.
4. $\tan 125° = -\cot 35°$.
5. $\cot 91° = -\tan 1°$.
6. $\sec 110° = -\csc 20°$.
7. $\csc 157° = \csc 23°$.
8. $\sin 204° = -\sin 24°$.
9. $\cos 359° = \cos 1°$.
10. $\tan 300° = -\cot 30°$.
11. $\cot 264° = \tan 6°$.
12. $\sec 244° = -\csc 26°$.
13. $\csc 271° = -\sec 1°$.
14. $\sin 163° 49' = \sin 16° 11'$.
15. $\cos 195° 33' = -\cos 15° 33'$.
16. $\tan 269° 15' = \cot 0° 45'$.
17. $\cot 139° 17' = -\cot 40° 43'$.
18. $\sec 299° 45' = \csc 29° 45'$.
19. $\csc 92° 25' = \sec 2° 25'$.

20. $\sin(-75^\circ) = -\sin 75^\circ = -\cos 15^\circ$, $\cos(-75^\circ) = \cos 75^\circ = \sin 15^\circ$, etc.

21. $\sin(-127^\circ) = -\sin 127^\circ = -\cos 37^\circ$, $\cos(-127^\circ) = \cos 127^\circ = -\sin 37^\circ$, etc.

22. $\sin(-200^\circ) = \sin 160^\circ = \sin 20^\circ$, $\cos(-200^\circ) = \cos 200^\circ = -\cos 20^\circ$, etc.

23. $\sin(-345^\circ) = -\sin 345^\circ = \sin 15^\circ$, $\cos(-345^\circ) = \cos 345^\circ = \cos 15^\circ$. etc.

24. $\sin(-52^\circ 37') = -\sin 52^\circ 37' = -\cos 37^\circ 23'$,
$\cos(-52^\circ 37') = \cos 52^\circ 37' = \sin 37^\circ 23'$, etc.

25. $\sin(-196^\circ 54') = \sin 196^\circ 54' = \cos 16^\circ 54'$,
$\cos(-196^\circ 54') = \cos 196^\circ 54' = -\cos 16^\circ 54'$, etc.

26. $\sin 120^\circ = \frac{1}{2}\sqrt{3}$, $\cos 120^\circ = -\frac{1}{2}$, etc.

27. $\sin 135^\circ = +\frac{1}{2}\sqrt{2}$, $\cos 135^\circ = -\frac{1}{2}\sqrt{2}$, etc.

28. $\sin 150^\circ = +\frac{1}{2}$, $\cos 150^\circ = -\frac{1}{2}\sqrt{3}$, etc.

29. $\sin 210^\circ = -\frac{1}{2}$, $\cos 210^\circ = -\frac{1}{2}\sqrt{3}$, etc.

30. $\sin 225^\circ = -\frac{1}{2}\sqrt{2}$, $\cos 225^\circ = -\frac{1}{2}\sqrt{2}$, etc.

31. $\sin 240^\circ = -\frac{1}{2}\sqrt{3}$, $\cos 240^\circ = -\frac{1}{2}$, etc.

32. $\sin 300^\circ = -\frac{1}{2}\sqrt{3}$, $\cos 300^\circ = +\frac{1}{2}$, etc.

33. $\sin(-30^\circ) = -\frac{1}{2}$, $\cos(-30^\circ) = +\frac{1}{2}\sqrt{3}$, etc

34. $\sin(-225^\circ) = +\frac{1}{2}\sqrt{2}$, $\cos(-225^\circ) = -\frac{1}{2}\sqrt{2}$, etc.

35. $\cos x = -\frac{1}{2}\sqrt{2}$ or $-\sqrt{\frac{1}{2}}$, etc., $x = 225^\circ$.

36. $\tan x = -\sqrt{\frac{1}{3}}$, $\sin x = \frac{1}{2}$, $\cos x = -\frac{1}{2}\sqrt{3}$, $x = 150^\circ$.

37. $\sin 3540^\circ = \sin 300^\circ = -\sin 60^\circ = -\frac{1}{2}\sqrt{3}$, $\cos 3540^\circ = +\frac{1}{2}$, etc.

38. 210° and 330°; 120° and 300°.

39. 135°, 225°, and −225°; 150° and −30°.

40. 30°, 150°, 390°, and 510°.

41. $\sin 168^\circ$, $\cos 334^\circ$, $\tan 225^\circ$, $\cot 252^\circ$,
$\sin 349^\circ$, $\cos 240^\circ$, $\tan 64^\circ$, $\cot 177^\circ$.

42. 0.848. (Hint: $\tan 238^\circ = \tan 58^\circ$, $\sin 122^\circ = \sin 58^\circ$).

43. -1.952.

44. $(a - b)\sin x$.

45. $m \sin x \cos x$.

46. $(a - b)\cot x - (a + b)\tan x$.

47. $a^2 + b^2 + 2ab\cos x$.

48. 0.

49. $\cos x \sin y - \sin x \cos y$.

50. $\tan x$.

51. Positive between $x = 0^\circ$ and $x = 135^\circ$, and between $x = 315^\circ$ and $x = 360^\circ$; negative between $x = 135^\circ$ and $x = 315^\circ$.

52. Positive between $x = 45^\circ$ and $x = 225^\circ$; negative between $x = 0^\circ$ and $x = 45^\circ$, and between $x = 225^\circ$ and $x = 360^\circ$.

53. $\sin(x - 90^\circ) = -\cos x$, $\cos(x - 90^\circ) = \sin x$, etc.

54. $\sin(x - 180^\circ) = -\sin x$, $\cos(x - 180^\circ) = -\cos x$, etc.

Exercises 53 and 54 should be solved by drawing suitable figures, and employing a mode of proof similar to that used in § 28.

EXERCISE XI.

1. $\sin(x+y) = \frac{56}{65}$, $\cos(x+y) = \frac{33}{65}$.

3. $\sin(90^\circ + y) = \cos y$, $\cos(90 + y) = -\sin y$, etc.

4. $\sin(180 - y) = \sin y$, $\cos(180 - y) = -\cos y$, etc.

5. $\sin(180 + y) = -\sin y$, $\cos(180 + y) = -\cos y$, etc.

6. $\sin(270 - y) = -\cos y$, $\cos(270 - y) = -\sin y$, etc.

7. $\sin(270 + y) = -\cos y$, $\cos(270^\circ + y) = \sin y$, etc.

8. $\sin(360^\circ - y) = -\sin y$, $\cos(360 - y) = \cos y$, etc.

9. $\sin(360 + y) = \sin y$, $\cos(360^\circ + y) = \cos y$, etc.

10. $\sin(x - 90^\circ) = -\cos x$, $\cos(x - 90^\circ) = \sin x$, etc.

11. $\sin(x - 180^\circ) = -\sin x$, $\cos(x - 180^\circ) = -\cos x$, etc.

12. $\sin(x - 270^\circ) = \cos x$, $\cos(x - 270^\circ) = -\sin x$, etc.

13. $\sin(-y) = -\sin y$, $\cos(-y) = \cos y$, etc.

14. $\sin(45^\circ - y) = \frac{1}{2}\sqrt{2}(\cos y - \sin y)$, $\cos(45^\circ - y) = \frac{1}{2}\sqrt{2}(\cos y + \sin y)$, etc.

15. $\sin(45^\circ + y) = \frac{1}{2}\sqrt{2}(\cos y + \sin y)$, $\cos(45^\circ + y) = \frac{1}{2}\sqrt{2}(\cos y - \sin y)$, etc.

16 $\sin(30^\circ + y) = \frac{1}{2}(\cos y + \sqrt{3}\sin y)$, $\cos(30^\circ + y) = \frac{1}{2}(\sqrt{3}\cos y - \sin y)$, etc.

17. $\sin(60^\circ - y) = \frac{1}{2}(\sqrt{3}\cos y - \sin y)$, $\cos(60^\circ - y) = \frac{1}{2}(\cos y + \sqrt{3}\sin y)$, etc.

18. $3\sin x - 4\sin^3 x$. 19. $4\cos^3 x - 3\cos x$. 20. 0. 21. $\frac{1}{2}\sqrt{3}$.

22. $\sin\frac{1}{2}x = \sqrt{\frac{1 - 0.4\sqrt{6}}{2}} = 0.10051$; $\cos\frac{1}{2}x = \sqrt{\frac{1 + 0.4\sqrt{6}}{2}} = 0.99494$.

23. $\cos 2x = -\frac{1}{2}$, $\tan 2x = -\sqrt{3}$.

24. $\sin 22\tfrac{1}{2}^\circ = \frac{1}{2}\sqrt{2 - \sqrt{2}} = 0.3827$, $\cos 22\tfrac{1}{2}^\circ = \frac{1}{2}\sqrt{2 + \sqrt{2}} = 0.9239$.
$\tan 22\tfrac{1}{2}^\circ = \sqrt{2} - 1 = 0.4142$, $\cot 22\tfrac{1}{2}^\circ = \sqrt{2} + 1 = 2.4142$.

25. $\sin 15^\circ = \frac{1}{2}\sqrt{2 - \sqrt{3}} = 0.2588$, $\cos 15^\circ = \frac{1}{2}\sqrt{2 + \sqrt{3}} = 0.9659$.
$\tan 15^\circ = 2 - \sqrt{3} = 0.2679$, $\cot 15^\circ = 2 + \sqrt{3} = 3.7321$.

27–33. The truth of these equations is to be established by expressing the given functions in terms of the *same* function of the *same* angle. Thus, in Example 27,

$$\sin 2x = 2\sin x\cos x,$$

and $2\tan x = 2\frac{\sin x}{\cos x}$, $1 + \tan^2 x = \sec^2 x = \frac{1}{\cos^2 x}$.

By making these substitutions in the given equation its truth will be evident.

34. $\sin A + \sin B + \sin C = \sin A + \sin B + \sin[180 - (A + B)]$
$= \sin A + \sin B + \sin(A + B)$
$= 2\sin\frac{1}{2}(A + B)\cos\frac{1}{2}(A - B) + 2\sin\frac{1}{2}(A + B)\cos\frac{1}{2}(A + B)$
$= 2\sin\frac{1}{2}(A + B)[\cos\frac{1}{2}(A - B) + \cos\frac{1}{2}(A + B)]$
$= 4\sin\frac{1}{2}(A + B)\cos\frac{1}{2}A\cos\frac{1}{2}B$, (see §§ 34 and 35)
But $\cos\frac{1}{2}C = \cos[90^\circ - \frac{1}{2}(A + B)] = \sin\frac{1}{2}(A + B)$.
Therefore, $\sin A + \sin B + \sin C = 4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$.

35. Proof similar to that for 34.

36. $$\begin{aligned}\tan A+\tan B+\tan C &= \frac{\sin A\cos B}{\cos A\cos B}+\frac{\cos A\sin B}{\cos A\cos B}+\frac{\sin C}{\cos C}\\ &= \frac{\sin C}{\cos A\cos B}+\frac{\sin C}{\cos C} = \frac{\sin C\cos C+\cos A\cos B\sin C}{\cos A\cos B\cos C}\\ &= \frac{(\cos A\cos B+\cos C)\sin C}{\cos A\cos B\cos C} = \frac{[\cos A\cos B-\cos(A+B)]\sin C}{\cos A\cos B\cos C}\\ &= \frac{\sin A\sin B\sin C}{\cos A\cos B\cos C} = \tan A\tan B\tan C.\end{aligned}$$

37. Proof similar to that for 36.

38. $\frac{2}{\sin 2x}$.

39. $2\cot 2x$.

40. $\frac{\cos(x-y)}{\sin x\cos y}$.

41. $\frac{\cos(x+y)}{\sin x\cos y}$.

42. $\tan^2 x$.

43. $\frac{\cos(x-y)}{\cos x\cos y}$.

44. $\frac{\cos(x+y)}{\cos x\cos y}$.

45. $\frac{\cos(x-y)}{\sin x\sin y}$.

46. $\frac{\cos(x+y)}{\sin x\sin y}$.

47. $\tan x\tan y$.

Exercise XII.

1. If, for instance, $B=90°$, [25] becomes $\frac{a}{b}=\sin A$.

3. $a^2=b^2+c^2$, $a^2=b^2+c^2-2bc$, $a^2=b^2+c^2+2bc$.

6. 90° in each case.

7. (i.) $\frac{a-b}{a+b}=\tan(A-45°)$, and a right triangle.

(ii.) $a+b=(a-b)(2+\sqrt{3})$, an isosceles triangle with the angles 30°, 30°, 120°.

Exercise XIII.

9. 300.

10. $AB=59.564$ miles; $AC=54.285$ miles.

11. 4.6064 miles, 4.4494 miles, 3.7733 miles.

12. 4.1501 and 8.67.

13. 6.1433 miles and 8.7918 miles.

14. 8 and 5.4723.

15. $a=5$, $c=9.6592$.

16. $a=7$, $b=8.573$.

17. Sides, 600 feet and 1039.2 feet; altitude, 519.6 feet.

18. 855 : 1607.

19. 5.438 and 6.857.

20. 15.588.

Exercise XIV.

11. 420.

12. The other diagonal = 124.617

Exercise XV.

11. 6.
12. 10.392.
14. 8.9212.
15. 25.
16. 3800 yards.
17. 729.7 yards.
18. 10.266.
19. $a = 5.0032$, $b = 2.3385$.
20. $26^\circ 0' 10''$ and $14^\circ 5' 50''$.

Exercise XVI.

11. $A = 36^\circ 52' 12''$, $B = 53^\circ 7' 48''$, $C = 90^\circ$.
12. $A = B = 33^\circ 33' 27''$, $C = 112^\circ 53' 6''$.
13. $A = B = C = 60^\circ$.
14. Impossible.
15. 15°, 45°, 120°.
16. 45°, 60°, 75°.
17. $4^\circ 23'$ W. of N., or W. of S.
18. 60°.
20. 0.88873.
21. 54.516 miles.

Exercise XVII.

1. 4333600.
2. 365.68.
3. 13260.
4. 8160.
5. 240.
6. 26208.
7. 15540.
8. 29450 or 6983.
9. $10\sqrt{3} = 17.3205$.
10. $6\sqrt{3} = 10.3923$.
11. 0.19952.
12. $ab \sin A$.
13. $\frac{1}{2}(a^2 - b^2) \tan A$.
14. 2421000.
15. 30°, 30°, 120°.

Exercise XVIII.

1. 21.166 miles; 24.966 miles.
2. 6.3399 miles.
3. 119.29 feet.
4. 30°.
5. 20 feet.
6. 2.6268 or 21.4704.
7. 276.14 yards.
8. 383.35 yards.

MISCELLANEOUS PROBLEMS.

2. 107 feet; 143 feet.
3. 1024 feet.
4. 37° 34′ 5″.
5. 238,400 miles.
6. 861,800 miles.
7. 2922.4 miles.
8. 60°.
9. 3.2.
10. 6.6.
11. 199.56 feet.
12. 43 feet.
13. 45 feet.
14. 26° 34′.
15. 78.37 feet.
16. 75 feet.
17. 1.44 miles.
19. 56.65 feet.
20. 69.28 feet.
21. 260 feet; 3690 feet.
22. 1.344 miles.
23. 235.8 yards.
27. 8 inches.
30. 460.45 feet.
31. 88.94 feet.
32. 13.657 miles.
34. 56.5 feet.
35. 51.6 feet.
36. 101.89 feet.
38. N. 76° 56′ E.; 13.94 miles an hour.
39. 442 yards.
40. 256 feet.
41. 3121 feet; 3633.5 feet.
42. 529.5 feet.
43. 41.41 feet.
44. 234.5 feet.
45. 25.43 miles.
46. 294.7 feet.
47. 12,492 feet.
48. 6.34 miles.
49. 210.44 feet.
51. 757.5 feet.
52. 520 yards.
53. 1366.4 feet.
54. 658 pounds; 22° 24′ with first force.
55. 88.33 pounds; 45° 37′ with known force.
58. 500.2; 536.3.
59. 345.47 feet.
60. 345.47 yards.
61. 61.23 feet.
63. 307.79.
64. 19.8; 35.7; 44.5.
65. 45°.
68. 60°.
69. 60°.
70. 30°.
73. $\frac{1}{2} bc \sin A$.
74. $\frac{1}{2} c^2 \sin A \sin B \csc (A + B)$.
75. $\sqrt{[s(s-a)(s-b)(s-c)]}$.

77. 199 A. 3 R. 10 P.
78. 210 A. 3 R. 26 P.
79. 12 A. 3 R. 37 P.
80. 3 A. 0 R. 6 P.
81. 12 A. 1 R. 14 P.
82. 4 A. 2 R. 26 P.
83. 14 A. 2 R. 9 P.
84. 61 A. 2 R.
85. 4 A. 2 R. 26 P.
86. 13.93, 23.21, 32.5 ch.
87. 9 A.
89. 876.31.
90. 1229.5.
92. 1075.3.
93. 2660.45.
94. 16,281.
95. 435.8 feet.
96. 49,089 feet.
97. 750 feet.
98. 422.4 feet.
99. 1835 feet.
100. 26.88.
103. 6.
108. 6.
110. 6087 feet.
111. 5° 25′ S.; 457.5 miles.
112. 460.8; 383.1 miles.
113. 229 miles; lat. 11° 39′ S.
114. S. 56° 7′ 30″ E.; 202.6 miles.
115. N. 17° 25′ W.; 37° 46′ N.
116. 56° 11′ E.; 244.3
121. Long. 68° 55′ W.
122. 103.6 miles.
124. 33° 18′ N.; 36° 24′ W.
125. N. 28° 47′ E.; 1293 miles.
126. S. 50° 40′ W.; 250.8; 20° 9′ W.
127. 38° 21′ N.; 55° 12′ W.

128. 171 miles; 32° 44′ W. 129. N. 36° 52′ W.; 36° 8′ W.

130. 173 miles; 51° 16′ S.; 34° 13′ E.

131. S. 50° 58′ E.; 47° 15′ N.; 20° 49′ W.

132. N. 53° 20′ E., 16° 7′ W.; *or* N. 53° 20′ W., 25° 53′ W.

133. N. 47° 42.5′ E., 19° 27′ N., 121° 51′ E.; *or* N. 47° 42.5′ W., 19° 27′ N., 116° 9′ E.; *or* S. 47° 42.5′ E., 14° 33′ N., 121° 48′ E.; *or* S 47° 42.5′ W., 14° 33′ N., 116° 12′ E.

137. N. 73′ E., 45 miles; 42° 15′ N., 69° 5′ W.

138 N. 72° W., 287 miles; 33° S., 13° 2′ E.

(11)
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ERICAL TRIGONOMETRY.

Exercise XIX.

2. 140°, 90°, 55°. 7. $\frac{8}{9}\pi$, 2π, $\frac{25}{9}\pi$

Exercise XX.

nust be equal to 90°. (ii.) If $a = 90°$, then $A = 90°$
$b = 90°$, then $B = 90°$, and $A = a$. (iii.) $c = 90°$
(iv.) $c = 90°$, $A = 90°$, $B = 90°$, $C = 90°$.

Exercise XXI.

e middle part = the product of the cotangents of
arts.
the middle part = the product of the sines of the

Exercise XXII.

, $B = 135° 42' 50''$, $C = 135° 34' 7''$.
, $a = 104° 53' 2''$, $b = 133° 39' 48''$.
, $B = 45°$.
, $B = 90°$.
mpossible; why?
, $c = 71° 27' 43''$, $A = 112° 57' 2''$.
$A = 35°$, $B = 65° 46' 7''$.
y?

Exercise XXIII.

$\frac{1}{2} b$, $\sin \frac{1}{2} B = \csc a \sin \frac{1}{2} b$, $\cos h = \cos a \sec \frac{1}{2} b$.
a, or $\cos A = \frac{1}{2} \cos a \sec^2 \frac{1}{2} a$.

3. $\sin \frac{1}{2} A = \sec \frac{1}{2} a \cos \frac{180°}{n}$, $\sin R = \sin \frac{1}{2} a \csc \frac{180°}{n}$,
$\sin r = \tan \frac{1}{2} a \cot \frac{180°}{n}$.

4. Tetrahedron, 70° 31′ 43″; octahedron, 109° 28′ 18″; icosahedron 138° 11′ 36″; cube, 90°; dodecahedron, 116° 33′ 44″.

5. $\cot \frac{1}{2} A = \sqrt{\cos a}$.

EXERCISE XXV.

1. (i.) $\tan m = \tan b \cos A$, $\cos a = \cos b \sec m \cos (c - m)$; (ii.) $\tan n = \tan a \cos B$, $\cos b = \cos a \sec n \cos (c - n)$.

EXERCISE XXVI.

1. (i.) $\cot x = \tan B \cos a$, $\cos A = \cos B \csc x \sin (C - x)$; (ii.) $\cot x = \tan A \cos b$, $\cos B = \cos A \csc x \sin (C - x)$.

EXERCISE XXXI.

4. $27.25''$.

EXERCISE XXXII.

1. If x denote the angle required, $\sin \frac{1}{2} x = \cos 18^\circ \sec 9^\circ$, $x = 148^\circ\, 42'$.

2. $\cos x = \cos A \cos B$.

3. Let $w =$ the inclination of the edge c to the plane of a and b. Then it is easily shown that $V = abc \sin l \sin w$. Now, conceive a sphere constructed having for centre the vertex of the trihedral angle whose edges are a, b, c. The spherical triangle, whose vertices are the points where a, b, c meet the surface of this sphere, has for its sides l, m, n; and w is equal to the perpendicular arc from the side l to the opposite vertex. Let L, M, N, denote the angles of this triangle. Then, by means of [38] and [47], we find that
$$\sin w = \sin m \sin N = 2 \sin m \sin \tfrac{1}{2} N \cos \tfrac{1}{2} N$$
$$= \frac{2}{\sin l} \sqrt{\sin s \sin (s - l) \sin (s - m) \sin (s - n)},$$
where $s = \frac{1}{2}(l + m + n)$;
hence, $V = 2abc\sqrt{\sin s \sin (s - l) \sin (s - m) \sin (s - n)}$.

4. (i.) 9,976,500 square miles; (ii.) 13,316,600 square miles.

5. Let $m =$ longitude of point where the ship crosses the equator, $b =$ her course at the equator, $d =$ distance sailed. Then
$$\tan m = \sin l \tan a, \quad \cos b = \cos l \sin a, \quad \cot d = \cot l \cos a.$$

6. Let $k =$ arc of the parallel between the places, $x =$ difference required; then $\sin \frac{1}{2} k = \sin \frac{1}{2} d \sec l$. $x = 90^\circ(\sqrt{2} - 1)$.

7. $\cos (m - m') = (\cos d - \sin l \sin l') \sec l \sec l'$; where m and m' are the longitudes of the places.

9. 44 min. past 12 o'clock.

10. 60°.

11. $\cos t = -\tan d \tan l$; time of sunrise $= 12 - \frac{t}{15}$ o'clock A.M.; time of sunset $= \frac{t}{15}$ o'clock P.M.; $\cos a = \sin d \sec l$. For longest day at Boston: time of sunrise, 4 hrs. 26 min. 50 sec. A.M.; time of sunset, 7 hrs. 33 min. 10 sec. P.M. Azimuth of sun at these times, 57° 25′ 15″; length of day, 15 hrs. 6 min. 20 sec.; for shortest day, times of sunrise and sunset are 7 hrs. 33 min. 10 sec. A.M. and 4 hrs. 26 min. 50 sec. P.M.; azimuth of sun, 122° 34′ 45″; length of day, 8 hrs. 53 min. 40 sec.

12. The problem is impossible when $\cot d < \tan l$; that is, for places in the frigid zone.

13. For the northern hemisphere and positive declination,
$\sin h = \sin l \sin d$, $\cot a = \cos l \tan d$.
Example: $h = 17° 14' 35''$, $a = 73° 51' 34''$ E.

14. The farther the place from the equator, the greater the sun's altitude at 6 A.M. in summer. At the equator it is 0°. At the north pole it is equal to the sun's declination. At a given place, the sun's altitude at 6 A.M. is a maximum on the longest day of the year, and then $\sin h = \sin l \sin e$ (where $e = 23° 27'$).

15. $\cos t = \cot l \tan d$. Times of bearing due east and due west are $12 - \frac{t}{15}$ o'clock A.M., and $\frac{t}{15}$ o'clock P.M., respectively.
Example: 6 hrs. 58 min. A.M. and 5 hrs. 2 min. P.M.

16. When the days and nights are equal, $d = 0°$, $\cos t = 0$, $t = 90°$; that is, sun is everywhere due east at 6 A.M., and due west at 6 P.M. Since l and d must both be less than 90°, $\cos t$ cannot be negative, therefore, t cannot be greater than 90°. As d increases, t decreases; that is, the times in question both approach noon. If $l < d$, then $\cos t > 1$; therefore this case is impossible. If $l = d$, then $\cos t = 1$, and $t = 0°$; that is, the times both coincide with noon. The explanation of this result is, that the sun at noon is in the zenith; hence on the prime vertical. At the pole $l = 90°$, $\cos t = 0$, $t = 90°$; therefore the sun in summer always bears due east at 6 A.M., and due west at 6 P.M.

17. $\sin l = \text{sid}\, d \csc h$.

18. 11° 50′ 35″.

19. The bearing of the wall, reckoned from the north point of the horizon, is given by the equation $\cot x = \cos l \tan d$; whence, for the given case, $x = 75° 12' 38''$.

20. 55° 45′ 6″ N.

21. 63° 23′ 41″ N. or S.

22. (i.) $\cos t = -\tan l \tan d$; (ii.) $t = z$; (iii.) the result is indeterminate.

23. $\cot a = \cos l \tan d$.

25. $h = 65° 37' 20''$.

26. $h = 58° 25' 15''$, $a = 152° 28'$.

27. $t = 45° 42'$, $l = 67° 58' 54''$.

28. $\sin d = \sin e \sin v$, $\tan r = \cos e \tan v$.

29. $d = 32° 24' 12''$, $r = 301° 48' 17''$.

30. $d = 20° 48' 12''$.

31. 3 hrs. 59 min. $27\frac{2}{3}$ sec. P.M.

32. $\cos \frac{1}{2} a = \sqrt{\cos \frac{1}{2}(l + h + p) \cos \frac{1}{2}(l + h - p) \sec l \sec h}$.

ANSWERS.

SURVEYING.

Exercise I.

1. 8 A. 64 P.
2. 29 A. $7\frac{3}{5}$ P.
3. 4 A. $5\frac{3}{25}$ P.
4. $115\frac{7}{20}$ P.
5. 3 A. 78 P.
6. 13 A. $6\frac{1}{10}$ P.
7. 11 A. 157 P.
8. 7.51925.
9. 13.0735.
10. 2 A. $58\frac{1}{2}$ P.
11. 4 A. 35 P.

Exercise II.

1. 2 A. 26 P.
2. 20 A. 12 P.
3. 2 A. 54 P.
4. 2 A. 151 P.
5. 8 A. 54 P.
6. 5 A. 42 P.
7. 2 A. 78 P.
8. 3 A. 122 P.
9. 6 A. 2 P.
10. 9 A. 40 P.

Exercise III.

1. 2 A. $12\frac{1}{2}$ P.
2. 98 A. 92 P.

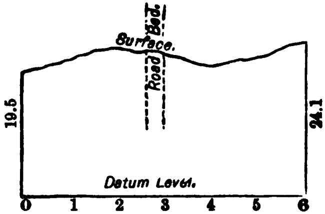
Exercise IV.

1. $AE = 3.75$ ch.
2. $AE = 3.50$ ch.; $EG = 3.42$ ch.
3. $AE = 4.55$ ch.
4. $AE = 5.50$ ch.
5. $CE = 4.455$ ch.
6. $AD = 2.275$ ch.; $BE = 1.82$ ch.
7. $AD = 4.51$ ch.; $BE = 3.608$ ch.
8. The distances on AB are 2, 3, and 5 ch.
9. EM (on EA) $= 2.5087$ ch. AN (on AB) $= 6.439$ ch.
10. Let $EG > DF$, then $\left\{\begin{array}{l} AE = 12.247 \text{ ch.} \\ AG = 9.798 \text{ ch.} \\ AD = 8.659 \text{ ch.} \\ AF = 6.928 \text{ ch.} \end{array}\right.$
11. Let $DG > EF$, then $\left\{\begin{array}{l} CG = 14.862 \text{ ch.} \\ CD = 13.113 \text{ ch.} \\ CF = 11.404 \text{ ch.} \\ CE = 10.062 \text{ ch.} \end{array}\right.$

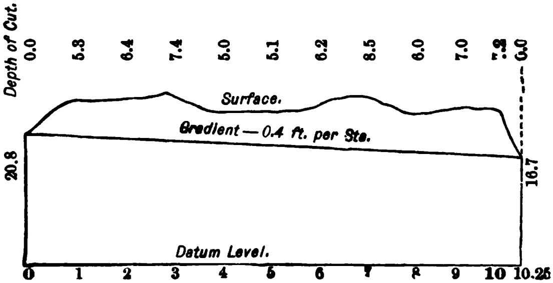
EXERCISE V.

1. 9.5 ft.

2. Third column: 26.944 opposite 0; 25.286 opposite 4.
Fifth column: 20, 19.5, 23, 22.3, 21.431, 20.4, 21.8, and 24.1.



3. Column *H.G.* 20.8, 20.4, 20.0, 19.6, etc.
Column *C.* 0.0, 5.3, 6.4, 7.4, 5.0, 5.1, etc.



NAVIGATION.

CHAPTER I.

DEFINITIONS. COURSE AND DISTANCE.

§ 1. Definitions.

Navigation is the art of conducting a ship from one port to another.

The **True Course** of a ship is the angle which her path makes with the true meridian.

The **Correct Magnetic Course** is the angle which the ship's path makes with the magnetic meridian.

When a ship is steered on one course, her path crosses the meridians at the same angle. Such a line is called the **Ship's Track, Rhumb-line,** or **Loxodromic Curve.**

The **Distance** is the path of the ship expressed in geographical miles.

The two most important problems of navigation are:

1. To find the present position of the ship.
2. To determine the future course.

The position (latitude and longitude) of a ship at sea may be found by two distinct methods, viz.:

1. By **Dead-reckoning,** which consists in keeping a record of the courses and distances sailed from a known point, and computing from these data the position arrived at.
2. By **Nautical Astronomy,** which consists in determining position by observations of the heavenly bodies.

When a ship is near land, the depth of water and the nature of the bottom, as determined by *sounding*, often serve to indicate her position.

§ 2. THE MARINER'S COMPASS.

When out of sight of land, vessels are steered by the **Mariner's Compass.**

The **Card** of the compass consists of a circular plate of mica covered with paper, the circumference of which is usually divided into degrees. An inner circle is divided into thirty-two equal parts by radial lines called *rhumb-lines*. The extremities of the rhumb-lines are called *points*, or *rhumbs*.

NOTE. A figure of the card of the compass, and a table giving the angular distance of each point, half-point, and quarter-point from the meridian, will be found on page 55 of Wentworth & Hill's Tables.

The principle upon which the points are named will be understood by inspecting a single quadrant. The point (N.E.) midway between N. and E. is named by writing these letters together. The point (N.N.E.) midway between N. and N.E. is named by joining these letters. The first point to the east of north is named N. by E., and the first point to the north of N.E. is named N.E. by N. In like manner, the points between E. and N.E. are named.

Naming the points in order is called **boxing the compass.**

One or more **Magnetic Needles** are attached to the lower surface of the card, parallel to the N.S. line, and having their N. and S. poles in the same direction as the corresponding points of the card.

The card turns upon a pivot in the centre of a cylindrical metallic box called the **Bowl,** which is supported by **Gimbals** in a square box.

In some compasses the card almost floats in alcohol, which relieves the pivot of the greater part of the weight of the card and needles.

The **Lubber-line** is a distinct vertical line on the inner side of the bowl.

It is often convenient to refer to the points of the compass as *right* or *left* of *north* or *south*. Suppose an observer to stand at the centre of the card, facing north. Then all points between N. and E. are right of north (R. of N.), and all points between N. and W. are left of north (L. of N.). In like manner, if the observer face south, all points between S. and W. are right of south (R. of S.), and all points between S. and E. are left of south (L. of S.).

Thus, W.S.W. is 6 points, or 67° 30′, R. of S.
N. by W. is 1 point, or 11° 15′, L. of N.
S.E. is 4 points, or 45°, L. of S.
E. by N. is 7 points, or 78° 45′, R. of N.

The compass is placed in a box called the **Binnacle.** When properly placed, the lubber-line is next to the bow of the vessel, and a line passing through the lubber-line and the centre of the card is parallel to the fore-and-aft line of the vessel.

The **Steering Compass** is placed immediately before the helmsman. The **Standard Compass** is commonly placed near the middle of the vessel, in such a position as to be least affected by the iron used in the construction of the vessel. (See § 5.)

§ 3. VARIATION OF THE NEEDLE.

This subject has been treated in "Surveying" (§ 23). The navigator is provided with charts which give the variation at different places.

§ 4. LOCAL ATTRACTION.

The disturbing effect upon the needle of iron outside of the ship (such as the iron in docks) is called local attraction.

§ 5. DEVIATION.

Deviation includes the changes in the direction of the needle caused by the iron in the ship.

Sub-permanent magnetism is the magnetism of the hard iron used in constructing the vessel. It depends upon the direction of the ship's head while building and the direction of the dipping needle at the place of construction.

Induced magnetism is the magnetism induced in the soft iron in the vessel by the earth's magnetic force.

Retentive magnetism is the temporary magnetism induced in an iron ship when her head is kept in one direction for some time.

Semicircular Deviation is caused by sub-permanent magnetism and the transient induced magnetism of vertical masses of soft iron in the ship. In wooden ships it disappears when the ship's correct magnetic course is N. or S., and is greatest when the correct magnetic course is E. or W. In iron ships the neutral points are in the direction of the ship's head and stern while building. The part due to sub-permanent magnetism remains the same in kind, but is different in amount, in all latitudes, unless the ship be strained or subjected to mechanical violence. The part produced by induced magnetism in vertical iron changes in different latitudes with the inclination of the dipping needle. It disappears at the magnetic equator, and is of contrary names on opposite sides of this equator.

Quadrantal Deviation is caused by the transient induced magnetism of horizontal masses of soft iron in the ship. It disappears when the correct magnetic course is N., S., E., or W., and is greatest when the correct magnetic course is N.E., N.W., S.E., or S.W. It remains the same in all magnetic latitudes, and is not appreciably affected by the lapse of time.

The amount of deviation may be found as follows:

1.. *By the known bearing of a distant object on shore.* The ship is swung around so as to bring its head successively to the thirty-two points of the compass, and the bearing of the object on shore, as given by the standard compass when the ship's head is steadied on each point, is recorded. The difference between the known correct magnetic bearing of the object and the successive bearings given by the standard compass will give the deviation for these points.

2. *By reciprocal bearings.* If there is no suitable object on land, a second compass may be placed on shore, and the mutual bearings of this instrument and the standard compass on shipboard, when the ship's head is steadied on a certain point, taken at the same instant by preconcerted signal. The bearing given by the compass on shore is then reversed. The difference between this reversed bearing and the bearing given by the standard compass will be the deviation for this point of the compass. In like manner, the deviation for each point may be found.

3. *By amplitudes and azimuths of a heavenly body.* (§ 32.)

The deviation is named as follows: If the correct magnetic bearing of the distant object (or the reversed bearing of the shore compass) is to the *right* of the reading of the standard compass, the deviation is *easterly;* and *vice versa.*

A **Deviation Table,** or deviation card, is a tabulated statement of the amount of deviation for each point of the compass, when the ship is on an even keel.

The **Heeling Error** must be estimated and allowed for separately.

Navigators commonly attempt to compensate for deviation by placing magnets and pieces of soft iron near the compass in such positions as to overcome the deviation.

Example 1. A ship is headed N.E. A distant object on shore bears S. 70° W., by the standard compass. The correct magnetic bearing of the object from the ship is known to be S. 89° W. Find the deviation.

Correct magnetic bearing	S. 89° W., or	89° R. of S.
Observed bearing	S. 70° W., or	70° R. of S.
Deviation		19° E.

The deviation is east, because 89° R. of S. is to the right of 70° R. of S.

Example 2. A ship is headed E. The bearing of the compass on shore from the ship, as given by the standard compass, is S. 21° W. The standard compass on the ship bears N. 8° W. from the compass on shore. Find the deviation.

Correct magnetic bearing (shore reversed)	S. 8° E., or	8° L. of S.
Observed bearing	S. 21° W., or	21° R. of S.
Deviation		29° W.

The deviation is west, because 8° L. of S. is to the left of 21° R. of S.

§ 6. The Wind.

The wind is named from the point of the compass from which it blows.

Windward is the direction from which the wind blows.

Leeward is the direction toward which the wind blows.

When the wind is directly behind a ship it is said to be **right aft**, or **right astern.** The ship is then sixteen points from the wind.

A ship is said to be on the **starboard tack** or the **port tack** according as the wind blows on the *right* or *left* side of the ship.

When the wind blows at right angles to the side of a ship, the ship is said to have the wind **on her beam.** The ship is then eight points from the wind.

When a ship is more than eight points from the wind, the wind is said to be **abaft the beam**; when the ship is less than eight points from the wind, the wind is said to be **before the beam.**

A ship is said to be **close-hauled** when she is as near the wind as she will lie; that is, when she is headed as near as possible toward the point from which the wind blows. Few ships can lie closer than five points from the wind.

§ 7. LEEWAY.

The angle which the fore-and-aft line of a vessel makes with **her** track is called **leeway.**

Let the arrow (Fig. 1) represent the direction of the wind, and *CB* the direction of the ship. Let *CB* represent the distance the ship would go in a given time if she were not allowed to move sideways, and *CD* the distance she would go in the same time if she were not allowed headway. By a principle of mechanics, the ship would, in the given time, describe *CA* the diagonal of the parallelogram of which *CB* and *CD* are adjacent sides.

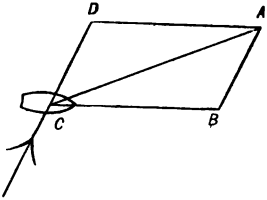


Fig. 1.

The angle *BCA* is the leeway.

The amount of leeway depends upon the trim of the ship, her velocity through the water, the sails set, etc.

§ 8. REDUCING THE COURSES.

The **Compass Course** is the course steered.

The true course is equal to the compass course corrected for variation, deviation, and leeway.

From the preceding explanations, the following rules for changing from one course to the other are evident:

To find the true course from the compass course:

Variation: { Allow easterly variation to the right.
Allow westerly variation to the left.

Deviation: { Allow easterly deviation to the right.
Allow westerly deviation to the left.

Leeway: { Allow leeway to the left on the starboard tack.
Allow leeway to the right on the port tack.

To find the compass course from the true course:

Variation: { Allow easterly variation to the left.
Allow westerly variation to the right.

Deviation: { Allow easterly deviation to the left.
Allow westerly deviation to the right.

Leeway: { Allow leeway to the right on the starboard tack.
Allow leeway to the left on the port tack.

EXAMPLE 1. The compass course is E.N.E., wind S.E., leeway 3 points, variation 10° E., deviation 2° 30′ E. Find the true course.

Since the wind is S.E., and the compass course E.N.E., the ship is on the starboard tack; hence, leeway is allowed to the left.

Compass course	6 pts. R. of N.
Leeway	3 pts. L.
The compass course corrected for leeway	3 pts. R. of N.
	= 33° 45′ R. of N.
Variation and deviation (10° E. + 2° 30′ E.)	12° 30′ R.
True course	46° 15′ R. of N.
	= N. 46° 15′ E.

EXAMPLE 2. The compass course is W.S.W., wind S., leeway 2½ points, variation 30° E., deviation 18° W. Find the true course.

Compass course	6 pts. R. of S.
Leeway	2½ pts. R.
	8½ pts. R. of S.
	= 7½ pts. L. of N.
	= 84° 23′ L. of N.
Variation and deviation (30° E. − 18° W. = 12° E.)	12° 0′ R.
True course	72° 23′ L. of N.
	= N. 72° 23′ W.

EXAMPLE 3. The true course is N.E. by N., the vessel is on the port tack, leeway 10°, variation 1 point W., deviation 5° 30′ E. Find the compass course.

True course	33° 45′ R. of N.
Variation	11° 15′ R.
Correct magnetic course	45° 0′ R. of N.
Leeway	10° 0′ L.
	35° 0′ R. of N.
Deviation (for 35° R. of N. = N.E. by N. nearly) . .	5° 30′ L.
Compass course	29° 30′ R. of N. = N. 29° 30′ E.

Exercise I.

	Compass Course.	*Wind.*	*Leeway.*	*Variation.*	*Deviation.*	*True Course.*
1.	S.	E.S.E.	1¼ pts.	52° 0′ W.	2° 0′ E.	Required.
2.	W.N.W.	N.	3 "	42° 0′ E.	18° 30′ W.	Required.
3.	S.S.E. ½ E.	S.W. ½ S.	3½ "	2¼ pts. E.	1¼ pts. W.	Required.
4.	Required.	S. by W.	¾ "	10° 30′ E.	19° 0′ W.	S. 79° W.
5.	W. ¼ N.	N.N.W.	1¾ "	8° 30′ E.	15° 35′ E.	Required.
6.	E. ¼ N.	N.N.E.	2¼ "	13° 0′ W.	20° 0′ E.	Required.
7.	Required.	N. by W.	½ "	14° 0′ E.	19° 0′ E.	S. 85° E.
8.	W.	N.N.W.	1¼ "	18° 30′ E.	21° 0′ W.	Required.
9.	E. ½ S.	N.N.E. ½ E.	2½ "	21° 0′ E.	4° 0′ W.	Required.
10.	Required.	N. by W.	2¾ "	2 pts. W.	3½ pts. E.	E. by S. ¼ S.
11.	Required.	N.E.	3¼ "	2¾ pts. E.	1½ pts. E.	N. by W.
12.	Required.	S.S.W.	2½ "	2¾ pts. E.	¾ pts. E.	N.N.W.
13.	Required.	. . . . . .	0 "	7° 0′ W.	15° 0′ W.	S. 64° E.
14.	Required.	. . . . . .	0 "	6° 0′ E.	20° 0′ W.	N. 44° W.
15.	N. 65° W.	. . . . . .	0 "	10° 0′ E.	3° 0′ E.	Required.
16.	S. 15° W.	. . . . . .	0 "	6° 0′ W.	18° 0′ E.	Required.
17.	S. 18° E.	. . . . . .	0 "	25° 0′ E.	10° 0′ E.	Required.
18.	N. 30° E.	S. by W.	1¼ "	12° 0′ E.	10° 0′ W.	Required.

§ 9. The Log.

The rate at which a vessel moves through the water is commonly determined by the **Log.**

The **Common Log** consists of the log-chip (often called simply the log), the log-line, and the sand-glass.

The **Log-chip** (Fig. 2) is a flat piece of wood in the form of a sector, of about five inches radius, having its arc loaded with lead to make it float in an upright position.

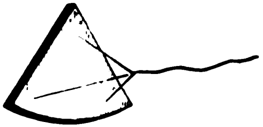


Fig. 2.

The **Log-line**, to which the log-chip is attached by three strings, is an ordinary cord about 150 fathoms long. The *stray-line* is the part of the log-line next the log-chip, and is commonly about two-thirds the length of the ship. The end of the stray-line is marked by a rag tied into the line. The remainder of the line is divided into equal parts called *knots*, and each knot is divided into tenths. Beginning at the end of the stray-line, the end of the first knot is marked by a piece of cord having one knot tied in it; the end of the second knot is marked by a cord with two knots tied in it; the end of the third knot is marked by a cord with three knots tied in it, etc.

A 30-seconds, 28-seconds, or 14-seconds sand-glass (Fig. 3) may be used.

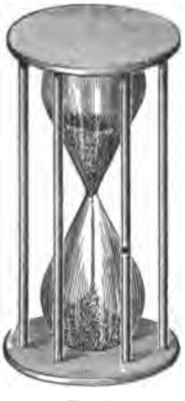


Fig. 3.

The knot is made the same part of a geographical mile (about 6080 ft.) that the time required for the sand to run through the glass is of an hour. The knot corresponding to a 30-seconds glass would be determined by the following proportion:

$$3600\text{ s.} : 30\text{ s.} :: 6080\text{ ft.} : x = 50\tfrac{2}{3}\text{ ft.}$$

As a vessel generally overruns her reckoning, the knot corresponding to a 30-seconds glass is made 50 ft. The proportion would give $47\frac{7}{24}$ ft. as the length of the knot corresponding to a 28-seconds glass, but in practice it is made 46 ft. 8 in.

The log-line is wound upon a *reel*. (Fig. 4.)

The operation of *heaving the log* is as follows: The officer throws the log-chip into the water on the lee side of the vessel.

It remains nearly stationary in the water while the line is reeled off as the vessel moves forward. The instant that the mark indicating the end of the stray-line passes over the side of the ship, the glass is turned. The operator then counts the number of knots run out during the time required for the sand to run through the glass. The line is given a quick jerk, which detaches the two lower of the three cords by which the log-chip is attached to the line, the log falls flat, and is readily drawn in.

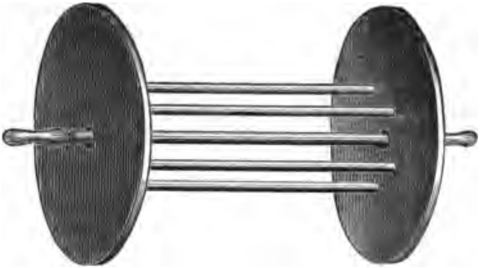


Fig. 4.

Since each knot is (approximately) the same part of a geographical mile that the time is of one hour, the number of knots run out, while the sand is running through the glass, indicates the rate of the vessel in geographical miles per hour.

The **Patent Harpoon Log,** one form of which is shown in Fig. 5, consists of a brass cylinder, which is towed through the water after the vessel by a line about fifty fathoms long. The cylinder consists of two parts. The hinder part, or rotator, is made to revolve by four curved blades as the log is drawn through the water; the forward part, which is kept from revolving by two straight blades, contains a registering apparatus, which is connected with the rotator, and shows the distance.

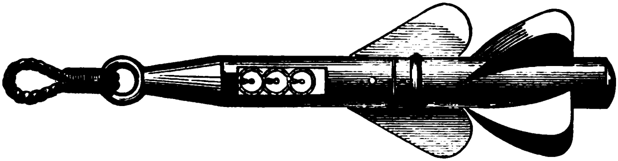


Fig. 5.

The **Patent Taffrail Log**, one form of which is shown in Fig. 6, consists of a registering apparatus which is permanently attached to the taffrail of the vessel, and a rotator which is towed through the water by a cord connected with the register. The revolutions of the rotator are communicated to the register by the cord.

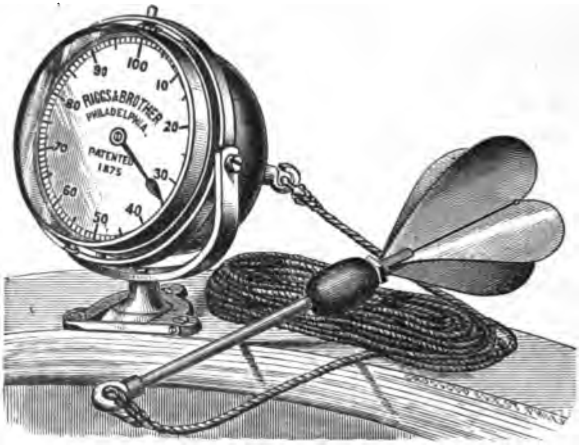


Fig. 6.

The **Ground Log** consists of the ordinary log-line with a hand-lead substituted for the log-chip. When it is hove, the lead sinks to the bottom and remains stationary. The line reeled off gives the speed *over the bottom.* This log is not affected by tides or currents. It may be used in shallow water.

The rate of paddle-steamers can be estimated from the number of revolutions of the paddle-wheels, quite as accurately as by the best patent log, especially in smooth water.

The Different Logs Compared.

The best results are obtained by the use of the harpoon log, *if it be in good order;* but the rapid motion of parts of the registering apparatus soon puts the best patent log out of order. Hence, fast ocean steamers ordinarily use the common log (hove every two hours) when in the open sea, and both the common and the patent log when nearing land.

The taffrail log is the most convenient, since the registering apparatus can be examined at any time without taking the rotator from the water; and if the rotator be lost it may be replaced at small cost, whereas, if the harpoon log be lost (a not infrequent occurrence), the cost of replacing it is considerable. But the taffrail log is not considered as reliable as the harpoon log, although it is growing in favor.

§ 10. CURRENTS.

The *set* of a current is its direction, or the point of the compass toward which it is moving.

The *drift* of a current is its rate in miles per hour.

The method of allowing for the effect of a current upon a ship will be explained hereafter (§ 18).

CHAPTER II.

THE SAILINGS.

§ 11. DEFINITIONS.

The **Difference of Latitude** of two places is the arc of a meridian included between their parallels.

The **Middle Latitude** of two places is the latitude of a parallel midway between them.

The **Difference of Longitude** of two places is the arc of the equator included between their meridians.

The **Departure** is the distance between the meridian left and the meridian arrived at, measured on a parallel.

§ 12. NOTATION.

C = the true course.
D = the distance in geographical miles.
L = latitude in general.
L' = the latitude left ("latitude from").
L'' = the latitude arrived at ("latitude in").
L_m = the middle latitude.
L_d = the difference of latitude.
λ = longitude in general.
λ' = the longitude left ("longitude from").
λ'' = the longitude arrived at ("longitude in").
λ_d = the difference of longitude.
p = the departure.

§ 13. Plane Sailing.

Case I. *When the distance is so small that the curvature of the earth may be neglected.*

Let CA (Fig. 7) be the distance sailed, CB the meridian through C, and AB the parallel through A.

Then, angle $ACB = C$,

$$CA = D,$$
$$CB = L_d,$$
$$AB = p.$$

By Plane Trigonometry,

$$\sin C = \frac{p}{D}, \quad \cos C = \frac{L_d}{D}, \quad \tan C = \frac{p}{L_d};$$
$$p = D \sin C, \quad L_d = D \cos C, \quad p = L_d \tan C;$$
$$D = \frac{p}{\sin C}, \quad D = \frac{L_d}{\cos C}, \quad L_d = \frac{p}{\tan C}.$$

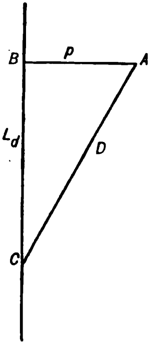


Fig. 7.

Case II. *When the distance is so great that the curvature of the earth cannot be neglected.*

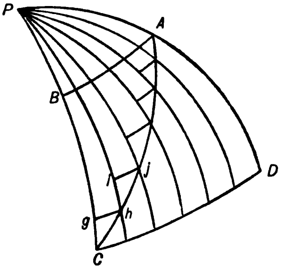


Fig. 8.

Let PCD (Fig. 8) represent a portion of the earth's surface, P the pole, CA the ship's track between C and A, PC and PD the meridians, and CD and BA the parallels through C and A.

Let CA be divided into parts Ch, hj, etc., so small that the curvature of the earth may be neglected for these distances; and let gh, ij, etc., be intercepts on the parallels corresponding to Ch, hj, etc.

Angle $PCA = PhA = PjA = \cdots\cdots = C$.

By Plane Trigonometry,

$$gh = Ch \sin C,$$
$$ij = hj \sin C,$$
$$\cdots\cdots = \cdots\cdots$$

$$\therefore gh + ij + \cdots\cdots = (Ch + hj + \cdots\cdots) \sin C.$$

Hence, $$p = D \sin C. \qquad (a)$$

Also,
$$\begin{aligned} Cg &= Ch \cos C, \\ hi &= hj \cos C, \\ \cdots\cdots &= \cdots\cdots \end{aligned}$$

$$\therefore Cg + hi + \cdots\cdots = (Ch + hj + \cdots\cdots) \cos C.$$

Hence, $$L_d = D \cos C. \qquad (b)$$

From (*a*) and (*b*),

$$\left.\begin{aligned} \sin C &= \frac{p}{D}, \\ p &= D \sin C, \\ D &= \frac{p}{\sin C}. \end{aligned}\right\} [1] \qquad \left.\begin{aligned} \cos C &= \frac{L_d}{D}, \\ L_d &= D \cos C, \\ D &= \frac{L_d}{\cos C}. \end{aligned}\right\} [2] \qquad \left.\begin{aligned} \tan C &= \frac{p}{L_d}, \\ p &= L_d \tan C, \\ L_d' &= \frac{p}{\tan C}. \end{aligned}\right\} [3]$$

These formulas are the same as in Case I.; but in Case II. p is evidently greater than BA and less than CD. (See § 15.)

Plane Sailing, as here treated, may be defined as the method of establishing the relations between course, distance, difference of latitude, and departure, on the supposition that small portions of the earth's surface may be regarded as plane.

The several problems are readily solved by formulas [1], [2], and [3].

In practice, the method by inspection is commonly used by navigators. A number of examples will be solved by this method in order to illustrate the use of the Traverse Tables.

Example 1. A ship sails from latitude 40° 20′ N., on a N.N.E. course for 92 miles. Find the departure and latitude in.

Solution I. *By computation.* The true course N.N.E. = N. 22° 30′ E.

To find the departure:

$$p = D \sin C. \qquad [1]$$

$D = 92.$

$C = 22° 30'$

$\log D \quad = 1.96379$

$\log \sin C = 9.58284$

$\log p = 1.54663$

$\therefore p = 35.2$ mi.

To find the difference of latitude:

$$L_d = D \cos C. \qquad [2]$$

$\log D \quad = 1.96379$

$\log \cos C \quad = 9.96562$

$\log L_d \quad = 1.92941$

$\therefore L_d = 85.0$ mi.

$= 1° 25'$ N.

$L \;= 40° 20'$ N.

$L' = 41° 45'$ N.

Solution II. *By inspection.* Since the hypotenuse and an acute angle of a right triangle are given to find the sides, the Traverse Table may be used. Corresponding to the course N.N.E. (2 points), and distance 92 miles, Table VIII.* gives latitude = 85, and departure = 35.2.

Example 2. A ship sails S.W. by W. 488 miles. Find the difference of latitude and departure.

By inspection. S.W. by W. = 5 points. In Table VIII., at the bottom of page 73, we find 5 points; hence, we look at the bottom of the page for the designations of the columns. The table only extends to a distance of 300; hence, we find the latitudes and departures corresponding to the distances 300 and 188, and take their sum.

For distance 300	latitude = 166.7	departure = 249.4
" " 188	" = 104.4	" = 156.3
" " 488	" = 271.1	" = 405.7

Or, we may divide the given distance by any convenient number, and find the latitude and departure corresponding to the quotient (the course remaining unchanged), then multiply each of these by the divisor of the distance.

488 ÷ 2 = 244.

For distance 244	latitude = 135.6	departure = 202.9
2	2	2
488	271.2	405.8

The results differ slightly by the two methods; but this small discrepancy is of no importance in practice. (See § 18.)

Example 3. Course 6½ points, distance 21.7; required the difference of latitude and departure.

By inspection. 6½ points is found at the bottom of page 67, Table VIII. Corresponding to distance 217 (21.7 × 10) the difference of latitude is 63.0, and the departure 207.7. Dividing each of these numbers by 10, the required difference of latitude is found to be 6.3, and the departure 20.8.

Example 4. If a ship run S.E. by S. from 1° 44′ north latitude, and is then by observation in 2° 46′ south latitude, what are the distance and departure?

S.E. by S. = 3 points. 1° 44′ N.
2° 46′ S.
4° 30′ = 270 mi. = difference of latitude.

---

* Wentworth & Hill's Tables.

On page 73, Table VIII., we find 3 points at the top; but the greatest difference of latitude on this page is 249; hence, we divide the given difference of latitude by 2. $270 \div 2 = 135$. The nearest difference of latitude to 135 is 134.7, opposite which the distance is 162, and the departure 90. Multiplying each of these numbers by 2, the required distance is 324 ($= 162 \times 2$), and the departure 180 ($= 90 \times 2$).

EXAMPLE 5. A ship sails N. 25° W. until the departure is 98 miles. Find the distance and difference of latitude.

Since the course is given in *degrees*, we turn to Table IX. At the top of page 102 we find 25°, and opposite departure 98, we find the required distance 232, and difference of latitude 210.3.

EXAMPLE 6. Find the course and distance corresponding to a difference of latitude of 696 miles, and a departure of 186 miles, the course lying between N. and E.

SOLUTION I. *By computation.*

To find the course:

$$\tan C = \frac{p}{L_d}. \qquad [3]$$

$$\begin{array}{l|l} p = 186 & \log p = 2.26951 \\ L_d = 696 & \operatorname{colog} L_d = \underline{7.15739} \\ & \log \tan C = 9.42690 \\ & \therefore C = \text{N. } 14^\circ\, 58' \text{ E.} \end{array}$$

To find the distance:

$$D = \frac{p}{\sin C}. \qquad [1]$$

$$\begin{array}{l} \log p = 2.26951 \\ \operatorname{colog} \sin C = \underline{0.58795} \\ \log D = 2.85746 \\ \therefore D = 720.2 \end{array}$$

SOLUTION II. *By inspection.* Since the given difference of latitude exceeds the limits of the tables, we divide each of the given numbers by 3; $696 \div 3 = 232$, $186 \div 3 = 62$. If the course be desired in degrees, we now turn to Table IX. and look for a difference of latitude of 232 opposite a departure of 62. On page 92, the nearest approximation is found. At the top of this page the angle is 15°; hence, the required course is N. 15° E. If the course be desired in points, we turn to Table VIII. The nearest approximation is found on page 66, at the top of which stands 1¼ points. Hence, the required course is N. by E. ¼ E. The distance is 720 ($= 240 \times 3$) miles.

NOTE. The answers, given at the end of this volume, to the exercises of this chapter, are obtained by computation. The student should, however, solve both by computation and inspection.

EXERCISE II.

	L'.	L''.	C.	D.	p.
1.	49° 57′ N.	Required.	S.W. by W.	488.0	Required.
2.	1° 45′ N.	Required.	S.E. by E.	487.8	Required.
3.	3° 15′ S.	Required.	N.E. by E. $\frac{3}{4}$ E.	449.1	Required.
4.	2° 10′ S.	Required.	N. by E.	267.0	Required.
5.	41° 30′ N.	Required.	S.S.W.	295.5	Required.
6.	21° 59′ S.	24° 49′ S.	Required.	360.0	Required.
7.	2° 9′ S.	3° 11′ N.	Required.	354.0	Required.
8.	1° 30′ N.	0° 26′ S.	S. by W.	Required.	Required.
9.	40° 17′ N.	37° 6′ N.	S. by W. $\frac{1}{2}$ W.	Required.	Required.
10.	38° 0′ N.	Required.	S.W. by W.	Required.	48.2
11.	18° 25′ N.	Required.	S.W. by W. $\frac{3}{4}$ W.	Required.	65.1
12.	50° 18′ N.	54° 48′ N.	Required.	299.0	Required.
13.	32° 30′ N.	19° 59′ N.	Required.	812.0	Required.
14.	2° 8′ S.	Required.	N. 11° E.	500.0	Required.
15.	20° 21′ S.	Required.	N. 20° E.	402.0	Required.
16.	40° 25′ S.	Required.	N. 87° E.	240.0	Required.
17.	20° 48′ N.	17° 13′ N.	Required.	Required.	289.2 W.
18.	51° 45′ N.	53° 11′ N.	Required.	Required.	128.0 E.
19.	0° 20′ S.	0° 18′ N.	Required.	Required.	142.7 E.
20.	40° 20′ N.	41° 37′ N.	Required.	Required.	52.6 W.

§ 14. PARALLEL SAILING.

Let P (Fig. 9) represent the pole, O the centre of the earth, A and B two places on the same parallel AB, PAE and PBQ meridians through A and B intercepting EQ on the equator. Let AD and BD be perpendicular to PO. Join AO. AB is the departure or meridian distance. Regard the earth as spherical.

$$AE = BQ = EOA = OAD$$
$$= \text{latitude of } A \text{ and } B.$$

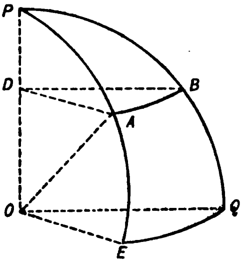


Fig. 9.

In the right triangle AOD,

$$\cos OAD = \frac{AD}{AO} = \frac{AD}{R}, \text{ or } \cos L = \frac{AD}{R}.$$
$$\therefore AD = R \cos L.$$

By Geometry, $EQ : AB :: EO : AD$
or $\lambda_o : p :: R : R \cos L.$

$$\left.\begin{aligned} \therefore p &= \lambda_o \cos L \\ \text{And} \quad \lambda_o &= p \sec L \end{aligned}\right\} \qquad [4]$$

Since $p = \lambda_o \cos L$, λ_o is the hypotenuse of the right triangle of which L is an acute angle and p the side adjacent, as in Fig. 10.

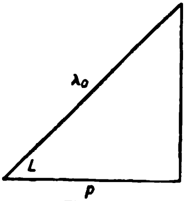


Fig. 10.

Since $p = \lambda_o \cos L$, the meridian distance varies as the cosine of the latitude.

Let $\lambda_o = 1° = 60' = 60$ miles, then the above formula becomes $p = 60 \cos L$.

Hence, to find the number of geographical miles in 1° of longitude at any place, multiply 60 by the cosine of the latitude.

The number of statute miles in a degree of longitude may be found by multiplying the numbers in the table by $\frac{69.16}{60}$, or by using the formula $p = 69.16 \cos L$.

Table showing the Number of Geographical Miles in a Degree of Longitude in every Latitude.

L.	Miles.	L.	Miles.	L.	Miles.	L.	Miles.	L.	Miles.	L.	Miles.
1°	59.99	16°	57.67	31°	51.43	46°	41.68	61°	29.09	76°	14.51
2°	59.96	17°	57.38	32°	50.88	47°	40.92	62°	28.17	77°	13.50
3°	59.92	18°	57.06	33°	50.32	48°	40.15	63°	27.24	78°	12.48
4°	59.85	19°	56.73	34°	49.74	49°	39.36	64°	26.30	79°	11.45
5°	59.77	20°	56.38	35°	49.15	50°	38.57	65°	25.36	80°	10.42
6°	59.67	21°	56.01	36°	48.54	51°	37.76	66°	24.41	81°	9.38
7°	59.56	22°	55.63	37°	47.92	52°	36.94	67°	23.44	82°	8.35
8°	59.42	23°	55.23	38°	47.28	53°	36.11	68°	22.48	83°	7.32
9°	59.26	24°	54.81	39°	46.63	54°	35.27	69°	21.50	84°	6.28
10°	59.09	25°	54.38	40°	45.96	55°	34.41	70°	20.52	85°	5.23
11°	58.89	26°	53.93	41°	45.28	56°	33.55	71°	19.53	86°	4.18
12°	58.69	27°	53.46	42°	44.59	57°	32.68	72°	18.54	87°	3.14
13°	58.46	28°	52.97	43°	43.88	58°	31.79	73°	17.54	88°	2.09
14°	58.22	29°	52.47	44°	43.16	59°	30.90	74°	16.54	89°	1.05
15°	57.95	30°	51.96	45°	42.43	60°	30.00	75°	15.53	90°	0.00

EXAMPLE 1. A ship in latitude 42° changes her longitude 3° 20′ by sailing on this parallel. Find her departure.

SOLUTION I.

$$\mathrm{p} = \lambda_d \cos \mathrm{L}. \qquad [4]$$

$p = 60 \times 3\frac{1}{3} \times \cos 42°$
$= 60 \times 3\frac{1}{3} \times .7431$
$= 148.6$ miles.

SOLUTION II. The table gives for $L = 42°$,

$1°$ of $\lambda = 44.59$.
$\therefore 3\frac{1}{3}°$ of $\lambda = 44.59 \times 3\frac{1}{3}$
$= 148.6$ miles.

SOLUTION III. By referring to Fig. 10, it will be seen that the given latitude corresponds to course, and difference of longitude to distance. Hence, p is the latitude in the Traverse Table corresponding to course 42° and distance $60 \times 3\frac{1}{3} = 200$, which gives 148.6 miles, as before.

EXAMPLE 2. A ship sails 176.2 miles due west from the Lizard, in latitude 49° 57′ N., and longitude 5° 14′ W. Find the longitude in.

SOLUTION I.

$$\lambda_d = \frac{\mathrm{p}}{\cos \mathrm{L}}. \qquad [4]$$

$p = 176.2$
$L = 49° 57'$

$\lambda' = 5° 14'$ W.
$\lambda_d = 4° 34'$ W.

$\lambda'' = 9° 48'$ W.

$\log p = 2.24601$
$\operatorname{colog} \cos L = 0.19148$

$\log \lambda_d = 2.43749$
$\therefore \lambda_d = 273.8$ mi.
$= 4° 34'$ W.

SOLUTION II.

From the table,

$1° \ \lambda$ for $49° \ L = 39.36$
$1° \ \lambda$ for $50° \ L = 38.57$

By interpolation,

$1° \ \lambda$ for $49° 57' \ L = 38.61$
$176.2 \div 38.61 = 4.56$
$\lambda_d = 4.56° = 4° 34'$.

EXERCISE III.

	L.	p.	λ'.	λ''.
1.	55° 55′	Required.	2° 10′ W.	12° 52′ E.
2.	52° 00′	Required.	0° 59′ W.	2° 24′ E.
3.	61° 25′	Required.	179° 20′ W.	176° 52′ E.
4.	56° 00′	Required.	3° 12′ W.	4° 8′ E.
5.	80° 00′	Required.	10° 0′ W.	17° 41′ W.
6.	60° 00′	204.0 E.	160° 2′ E.	Required.
7.	51° 28′	70.9 E.	32° 7′ W.	Required.
8.	64° 16′	265.7 W.	170° 0′ W.	Required.
9.	46° 37′	352.0 E.	163° 42′ E.	Required.
10.	39° 57′	398.0 W.	4° 8′ W.	Required.

11. From latitude 32° 3′ S., longitude 179° 45′ W., a ship makes 54 miles west (true). Required the longitude in.

Note. In changing from the true to the compass course, apply variation and leeway, then deviation, which is supposed to be given for the course to which it is directly applied.

12. From latitude 35° 30′ S., longitude 27° 28′ W., a ship sails east (true) 301 miles. Required the longitude in and the compass course; variation 1¾ points E., leeway ¼ point to the left, deviation 8° 50′ E.

§ 15. Middle Latitude Sailing.

Middle Latitude Sailing is a combination of plane and parallel sailings, chiefly for the purpose of determining difference of longitude.

Let *CA* (Fig. 11) represent the ship's track, *CD* and *BA* parallels, and *PBC* and *PAD* meridians through *C* and *A* respectively. Let *EF* be a parallel midway between *C* and *A*.

It is shown in plane sailing that p is less than *CD* and greater than *BA*. It is nearly equal to *EF*, and exactly equal to an intercept a little nearer to the pole than *EF*.

Hence, $L_m = \frac{1}{2}(L' + L'')$, nearly, if both places are on the same side of the equator.

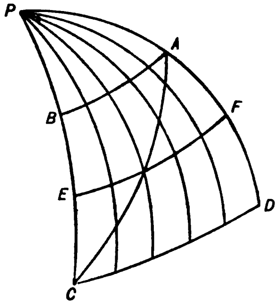


Fig. 11.

In middle latitude sailing p is first found as in plane sailing, then λ_d is found, on the supposition that λ_d is equal to the change of longitude which the ship would have made by sailing a distance p on a parallel in latitude L_m.

By parallel sailing,

$$p = \lambda_d \cos L_m, \quad \lambda_d = p \sec L_m. \qquad [5]$$

By plane sailing,

$$p = D \sin C, \quad \tan C = \frac{p}{L_d}.$$

Hence, by combining,

$$\tan C = \frac{\lambda_d \cos L_m}{L_d}, \qquad [6]$$

$$\lambda_d = D \sin C \sec L_m. \qquad [7]$$

The formula $L_m = \frac{1}{2}(L' + L'')$ may be used without sensible error in low latitudes when the course is not less than about 45°, and both places are on the same side of the equator.

When the places are on different sides of the equator, and the distance not great, $p = \lambda_d$ may be used; but when the distance is considerable, the following method is preferable.

Let CA (Fig. 12), represent the rhumb-line crossing the equator EQ at B. Let CE and AQ represent meridians.

For the rhumb CB the middle latitude is $\frac{1}{2}CE$; hence, EB (the difference of longitude of C and B) may be found. For the rhumb BA the middle latitude is $\frac{1}{2}AQ$; hence, BQ (the difference of longitude of B and A) may be found.

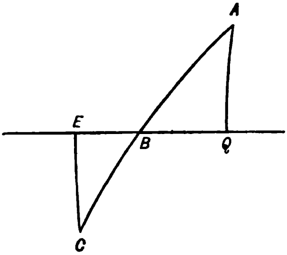


Fig. 12.

Then $\lambda_d = EB + BQ$.

In high latitudes, or in any latitude, if the course be less than about 45° and the distance great,

$$L_m = \frac{1}{2}(L' + L'') + d,$$

d being a small arc. This correction of the middle latitude is seldom used in practice.

The relations of the elements of middle latitude sailing may be represented by combining the triangles of plane and parallel sailings, as in Fig. 13.

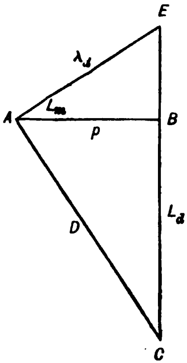


Fig. 13.

$ACB = C$, $AB = p$,
$CA = D$, $AE = \lambda_d$,
$CB = L_d$, $BAE = L_m$.

EXAMPLE 1. A ship sails from Sandy Hook light, in latitude 40° 28′ N., longitude 74° W., on an E.S.E. course, 62 miles. Find the latitude and longitude in.

SOLUTION I. *By computation.*

To find the latitude in:

$$\mathbf{L}_d = \mathbf{D}\cos\mathbf{C}. \qquad [2]$$

$D = 62,\qquad \log D = 1.79239$

$C = 67^\circ\, 30',\qquad \log\cos C = \underline{9.58284}$

$\log L_d = 1.37523$

$\therefore L_d = 23.7 = 24'\ \mathrm{S}.$

$L' = 40^\circ\, 28'\ \mathrm{N}.$

$L_d = \underline{\ 24'\ \mathrm{S}.}$

$L'' = 40^\circ\ \ 4'\ \mathrm{N}.$

To find the middle latitude:

$L' = 40^\circ\, 28'\ \mathrm{N}.$

$L'' = \underline{40^\circ\ \ 4'\ \mathrm{N}.}$

$2\overline{)80^\circ\, 32'}$

$L_m \doteq 40^\circ\, 16'$

To find the longitude in:

$$\lambda_d = \mathbf{D}\sin\mathbf{C}\sec\mathbf{L}_m. \qquad [7]$$

$\log D = 1.79239$

$\log\sin C = 9.96562$

$\log\sec L_m = \underline{0.11745}$

$\log\lambda_d = 1.87546$

$\therefore \lambda_d = 75.1$

$= 1^\circ\, 15'\ \mathrm{E}.$

$\lambda' = 74^\circ\ \ 0'\ \mathrm{W}.$

$\lambda_d = \underline{\ \ 1^\circ\, 15'\ \mathrm{E}.}$

$\lambda'' = 72^\circ\, 45'\ \mathrm{W}.$

SOLUTION II. *By inspection.* In Fig. 13, it is evident that if we enter the Traverse Table with the course and distance, the difference of latitude and departure may be found as in plane sailing, p. 17, Ex. 1. Corresponding to the course E.S.E. and distance 62, Table VIII., page 69, gives the difference of latitude 23.7, and the departure 57.3; whence, the latitude in and middle latitude may be found as in Solution I.

In Fig. 13, the difference of longitude AE is the distance corresponding to the course E and the departure AB. But the angle $E = 90^\circ - BAE$; that is, the angle E is the complement of the middle latitude. In this Example, the middle latitude is 40° 16′; whence, the co-middle latitude is $90^\circ - 40^\circ\, 16' = 49^\circ\, 44' = 50^\circ$ approximately. Corresponding to the course 50° and the departure 57.3 (the nearest is 57.5), Table IX., page 117, gives the distance 75, which is the difference of longitude required; whence, the longitude in may be found as in Solution I.

EXAMPLE 2. A ship sails from latitude 20° N., longitude 160° W. until the difference of latitude is 1° 50′ S., and the departure 440.2 miles W. Find the latitude in, the longitude in, the course, and distance, by inspection.

$$\begin{array}{ll} L' & = 20^\circ\ 0'\ \text{N.} \\ L_d & = \underline{\ \ 1^\circ\ 50'\ \text{S.}} \\ L'' & = 18^\circ\ 10'\ \text{N.} \end{array}$$

$$\begin{array}{rl} L' & = 20^\circ\ 0'\ \text{N.} \\ L'' & = \underline{18^\circ\ 10'\ \text{N.}} \\ & 2\overline{)38^\circ\ 10'} \\ L_m & = 19^\circ\ 5' \\ \text{co}\ L_m & = 70^\circ\ 55' \end{array}$$

$$\begin{array}{ll} \lambda' & = 160^\circ\ 0'\ \text{W.} \\ \lambda_d & = \underline{\ \ \ \ 7^\circ\ 45'\ \text{W.}} \\ \lambda'' & = 167^\circ\ 45'\ \text{W.} \end{array}$$

To find the longitude in. Since 440.2 exceeds the limits of the table, one-third of this number is taken. $440.2 \div 3 = 146.7$. Corresponding to course 71° (= co L_m) and departure 146.7, Table XI., page 96, gives the distance 155. $155 \times 3 = 465 = 7^\circ\ 45'$, which is the difference of longitude; whence, the longitude in is found to be 167° 45′ W.

To find the course and distance. Since the departure 440.2 exceeds the difference of latitude 110 (= 1° 50′), the course and designations of the columns must be sought at the bottom of the page. $110 \div 2 = 55$, and $440.2 \div 2 = 220.1$. Corresponding to the difference of latitude 55 and the departure 220.1 (the nearest are 54.9 and 220.3), Table XI., page 91, gives the distance 227. $227 \times 2 = 454$. At the bottom of the page, the course 76° is found. Hence, the course and distance required are S. 76° W. 454 miles.

EXAMPLE 3. A navigator wishes to sail from the Lizard, in latitude 49° 57′ N., longitude 5° 14′ W., to St. Mary's Island, in latitude 37° N. and longitude 25° 6′ W. Find the true course and distance.

To find L_d:

$$\begin{array}{ll} L' & = 49^\circ\ 57'\ \text{N.} \\ L'' & = \underline{37^\circ\ \ 0'\ \text{N.}} \\ L_d & = 12^\circ\ 57'\ \text{S.} \\ & = 777\ \text{miles.} \end{array}$$

To find L_m:

$$\begin{array}{rl} L' & = 49^\circ\ 57'\ \text{N.} \\ L'' & = \underline{37^\circ\ \ 0'\ \text{N.}} \\ & 2\overline{)86^\circ\ 57'} \\ L_m & = 43^\circ\ 29' \end{array}$$

To find λ_d:

$$\begin{array}{ll} \lambda' & = \ \ 5^\circ\ 14'\ \text{W.} \\ \lambda'' & = \underline{25^\circ\ \ 6'\ \text{W.}} \\ \lambda_d & = 19^\circ\ 52'\ \text{W.} \\ & = 1192\ \text{miles.} \end{array}$$

To find the course:

$$\tan C = \frac{\lambda_d \cos L_m}{L_d}. \qquad [6]$$

$$\begin{array}{lll} \lambda_d = 1192, & \log \lambda_d & = 3.07628 \\ L_m = 43^\circ\ 29', & \log \cos L_m & = 9.86068 \\ L_d = 777, & \text{colog}\, L_d & = \underline{7.10958} \\ & \tan C & = 10.04654 \\ & & \therefore C = \text{S.}\ 48^\circ\ 4'\ \text{W.} \end{array}$$

To find the distance:

$$D = L_d \sec C. \qquad [2]$$

$$\begin{array}{ll} \log L_d & = 2.89042 \\ \log \sec C & = \underline{0.17505} \\ \log D & = 3.06547 \\ & \therefore D = 1163\ \text{miles.} \end{array}$$

EXERCISE IV.

	l'.	l''.	λ'.	λ''.	C.	D.
1.	25° 35′ N.	27° 28′ N.	60° 0′ W.	54° 55′ W.	Required.	Required.
2.	32° 30′ N.	34° 10′ N.	25° 24′ W.	29° 8′ W.	Required.	Required.
3.	39° 30′ S.	41° 0′ S.	74° 20′ E.	70° 12′ E.	Required.	Required.
4.	46° 24′ S.	Required.	178° 28′ E.	Required.	S.E. ¾ E.	278.0
5.	20° 29′ N.	Required.	179° 10′ W.	Required.	W. by S. ½ S.	333.0
6.	0° 56′ N.	Required.	29° 50′ W.	Required.	S. 47° E.	168.0
7.	42° 25′ N.	Required.	66° 14′ W.	Required.	S.E. by E.	25.0
8.	42° 8′ N.	Required.	65° 48′ W.	Required.	E. ½ S.	126.0
9.	41° 52′ N.	Required.	62° 47′ W.	Required.	E. ½ S.	161.0
10.	41° 38′ N.	41° 26′ N.	59° 16′ W.	Required.	E. by S.	Required.
11.	41° 19′ N.	41° 11′ N.	57° 47′ W.	Required.	Required.	167.0
12.	46° 28′ N.	45° 17′ N.	22° 18′ W.	19° 39′ W.	Required.	Required.
13.	25° 30′ S.	28° 15′ S.	2° 15′ E.	11° 17′ E.	Required.	Required.
14.	33° 40′ N.	30° 49′ N.	13° 20′ E.	17° 56′ E.	Required.	Required.
15.	19° 30′ S.	17° 24′ S.	0° 10′ E.	1° 28′ W.	Required.	Required.

16. A ship sails from Boston light-house, in latitude 49° 20′ N., longitude 71° 4′ W., on a N.N.E. course, 184 miles. Find the latitude and longitude in.

17. A ship sails from Cape May, in latitude 38° 56′ N., longitude 74° 57′ W., on a S.S.E. course, 240 miles. Find the latitude and longitude in.

18. A ship sails from Cape Cod light, in latitude 42° 2′ N., longitude 70° 3′ W., on an E. by N. compass course, 170 miles; wind S.E. by S., leeway ¼ point, deviation 17¾° E., variation 11¼° W. Find the latitude and longitude in.

19. A ship sails from Cape Cod light on a S.S.E. compass course, 140 miles; deviation 5½° E., variation 11¼° W. Find the latitude and longitude in.

20. A ship sails from latitude 55° 1′ N., longitude 1° 25′ W. on a S.W. compass course, 101 miles; wind W.N.W., leeway 1¼ points, deviation 6° W., variation 24° 56′ W. Find the latitude and longitude in.

21. A ship sails from the Bermudas, in latitude 32° 18′ N., longitude 64° 50′ W., on a W.S.W. compass course, 190 miles; deviation 1 point W., variation 1 point W. Find the latitude and longitude in.

22. A ship sails from the Bermudas on a W.N.W. compass course, 90 miles; wind S.W., leeway 1 point, deviation 1 point E., variation 1 point W. Find latitude and longitude in.

23. A navigator wishes to sail on a rhumb from the Bermudas to Cape Fear, in latitude 33° 52′ N., longitude 78° W.; variation 10° W., deviation 7° W. Find the compass course and distance.

24. A ship from latitude 36° 32′ N. sails between south and west until she has made 480 miles of departure, and 9° 22′ of difference of longitude. Required the latitude in, the course steered, and the distance run. [Take $L_m = \frac{1}{2}(L' + L'') + 13'$.]

§ 16. MERCATOR'S SAILING.

Mercator's Chart.

Mercator's chart represents the surface of the earth as a rectangle. The meridians are represented by straight lines perpendicular to a straight line representing the equator, and the parallels are represented by straight lines parallel to the line representing the equator. On this chart, any two meridians intercept equal distances on all the parallels; but on the earth, the intercept on any parallel by two meridians is less than the intercept on the equator.

Hence, at no place except the equator does this chart correctly represent the intercepts on the parallels, and the error increases as we approach the poles.

In order to compensate for this error, the distance of each parallel from the equator is proportionally increased.

These augmented latitudes are called **Meridional Parts,** and are given for every latitude in Table XI.

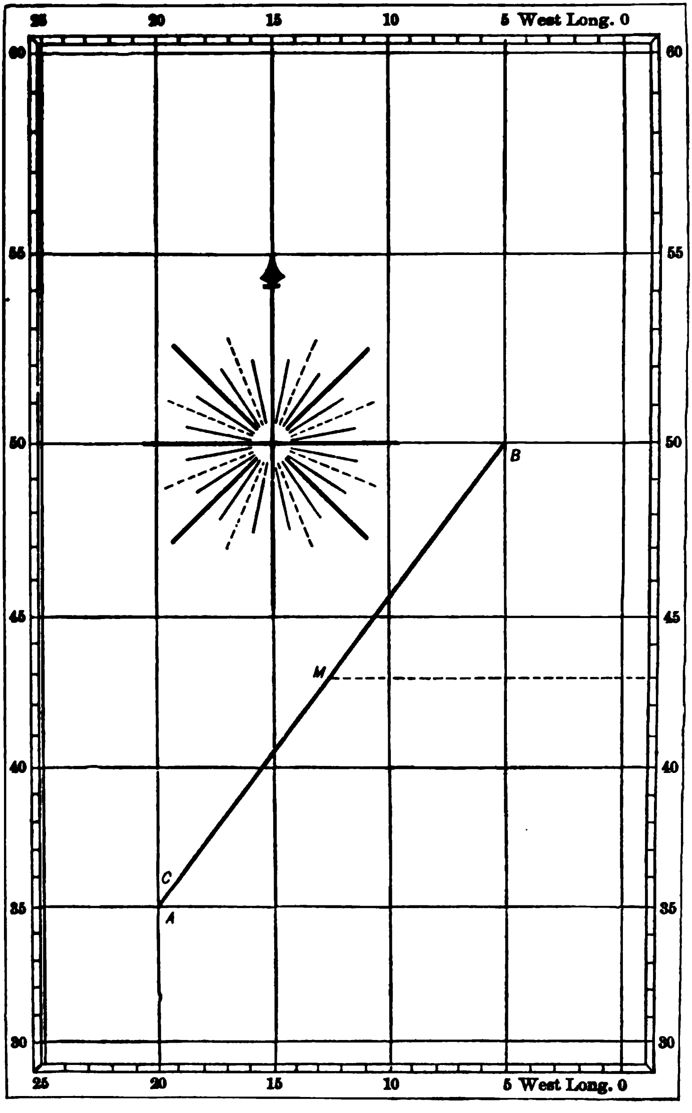

5 West Long. 0
B
M
C
A
5 West Long. 0

Fig. 14.

On account of the spheroidal form of the earth, the development of the formula for the computation of this table is too difficult for presentation in an elementary work.

The *meridional difference of latitude* is the expanded arc of the meridian between two parallels. It may be found by taking the difference or sum of the meridional parts corresponding to the two latitudes, according as the latitudes are of the same name or of contrary names.

If the earth be regarded as a sphere,

$$\lambda_d = \text{p} \sec \text{L}. \qquad [4]$$

But on Mercator's chart, $p = \lambda_d$; that is, p is too great in the ratio of the secant of the latitude. Hence, the intercepts of the parallels on the meridians must be increased in the same ratio. The augmented latitude for n degrees of latitude would be

$$1' (\sec 1' + \sec 2' + \sec 3' + \cdots + \sec n').$$

The construction and use of Mercator's chart will be understood by referring to Fig. 14, which represents a chart extending from 30° to 60° north latitude, and from 0° to 25° west longitude.

The horizontal line representing the parallel of 30° is first drawn, and divided into equal parts. Through the points of division the meridians are drawn perpendicular to the parallel. In this case each division of the parallel represents $5° = 300'$.

From the Table of Meridional Parts,

Meridional parts corresponding to a latitude of 30° = 1876.9′
Meridional parts corresponding to a latitude of 35° = 2231.1′
Meridional difference of latitude for 5° . . . . = 354.2′

Hence, the parallel of 35° is drawn at a distance of 354.2′ above the parallel of 30°.

Meridional parts corresponding to a latitude of 35° = 2231.1′
Meridional parts corresponding to a latitude of 40° = 2607.9′
Meridional difference of latitude for 5° . . . . = 376.8′

Hence, the parallel of 40° is drawn at a distance of 376.8′ above the parallel of 35°, etc.

Let it be required to find from the chart the course and distance by rhumb-line from *A*, in latitude 35° N., longitude 20° W., to *B*, in latitude 50° N., longitude 5° W.

A is at the intersection of the meridian of 20° and the parallel of 35°; *B* is at the intersection of the meridian of 5° and the parallel of 50°. Draw *AB*. *AB* represents the rhumb-line from *A* to *B*, for it makes the same angle (*C*) with all of the meridians.

The course *C* may be found by making one edge of a pair of parallel rulers coincide with *AB*, and extending until the other edge passes through the centre of the compass-rose above. The rhumb-line of the compass-rose which most nearly coincides with the edge of the ruler will show the course.

The distance may be found by carrying *m*, the middle point of *AB*, over to the side, and measuring up and down a distance equal to one-half of *AB*. The number of degrees between the extreme points reduced to miles will be the distance required.

It is evident that Mercator's chart shows relative *positions* accurately, but not relative *distances*.

Let *CA* (Fig. 15) represent the ship's track. *CB* will represent the true difference of latitude, and *AB* the departure. Take *CE* equal to the meridional difference of latitude corresponding to *CB*, and through *E* draw *ED* parallel to *BA* to meet *CA* produced in *D*. Then *CD* will represent the ship's track on Mercator's chart, and *ED* will represent the difference of longitude.

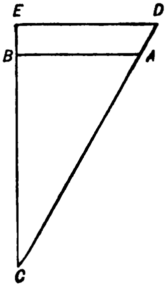


Fig. 15.

$CE =$ Mer. L_d

$= \text{meridional difference of latitude.}$

$CB = L_d. \qquad BA = p. \qquad ED = \lambda_d.$

In the triangle *CED*,

$$\tan C = \frac{ED}{EC},$$

or
$$\left.\begin{aligned}\tan C &= \frac{\lambda_d}{\text{Mer. } L_d},\\ \therefore \lambda_d &= \text{Mer. } L_d \times \tan C.\end{aligned}\right\} \qquad [8]$$

Also,
$$\tan C = \frac{p}{L_d}. \qquad [3]$$

$$\therefore \lambda_d = \frac{\text{Mer. } L_d \times p}{L_d}. \qquad [9]$$

Also,
$$D = L_d \sec C. \qquad [2]$$

When to use Middle Latitude and Mercator's Sailings.

Either Middle Latitude or Mercator's Sailing may be used when the distance is small (such as an ordinary day's run); but when the distance is great, Mercator's Sailing should be employed, if the course be less than about 45°; and Middle Latitude Sailing should be employed if the course exceed about 45°.

This distinction is of special importance in high latitudes.

EXAMPLE 1. A ship sails from latitude 40° 45′ N., longitude 74° W., E.N.E., 200 miles. Find latitude and longitude in.

SOLUTION I. *By computation.*

To find latitude in:

$$L_d = D \cos C. \qquad [2]$$

$D = 200$, $\log D = 2.30103$

$C = 67° 30'$, $\log \cos C = 9.58284$

$\log L_d = 1.88387$

$\therefore L_d = 76.5$ miles

$= 1° 17'$ N.

$L' = 40° 45'$ N.

$L'' = 42° \ 2'$ N.

To find longitude in:

$L' = 40° 45'$ N., Mer. Parts $= 2666.8$

$L'' = 42° \ 2'$ N., Mer. Parts $= 2769.0$

Mer. $L_d = 102.2$

$$\lambda_d = \text{Mer. } L_d \times \tan C. \qquad [8]$$

Mer. $L_d = 102.2$, $\log$ Mer. $L_d = 2.00945$

$C = 67° 30'$, $\log \tan C = 10.38278$

$\log \lambda_d = 2.39223$

$\therefore \lambda_d = 246.7$ miles

$= 4° \ 7'$ E.

$\lambda' = 74° \ 0'$ W.

$\lambda'' = 69° 53'$ W.

SOLUTION II. *By inspection.* E.N.E. = 6 points. Corresponding to a course of 6 points and a distance of 200, the true difference of latitude is found in Table VIII. to be 76.5; whence, the latitude in and the meridional difference of latitude are found as in Solution I. Corresponding to the course 6 points, and the meridional difference of latitude 102 as a difference of latitude, the same table gives the departure 246.7, which is equal to the difference of longitude required; whence, the longitude in may be found.

EXAMPLE 2. Required the course and distance from Halifax, in latitude 44° 40′ N. and longitude 63° 35′ W., to Ireland island, in latitude 32° 19′ N. and longitude 64° 49′ W.

$L' = 44° 40' \text{ N.}$ Mer. Parts $= 2985.6$ $\lambda' = 63° 35' \text{ W.}$
$L'' = 32° 19' \text{ N.}$ Mer. Parts $= 2038.6$ $\lambda'' = 64° 49' \text{ W.}$
$L_d = 12° 21' \text{ S.}$ Mer. $L_d = 947.0$ $\lambda_d = 1° 14' \text{ W.}$
$= 741$ miles. $= 74$ miles.

To find the course:

$$\tan C = \frac{\lambda_d}{\text{Mer. } L_d}. \qquad [8]$$

$\lambda_d = 74,$ $\log \lambda_d = 1.86923$
Mer. $L_d = 947,$ colog Mer. $L_d = 7.02365$
$\log \tan C = 8.89288$
$\therefore C = \text{S. } 4° 28' \text{ W.}$

To find the distance:

$$D = L_d \sec C. \qquad [2]$$

$L_d = 741,$ $\log L_d = 2.86982$
$C = 4° 28',$ $\log \sec C = 0.00132$
$\log D = 2.87114$
$\therefore D = 743.3$ miles.

EXERCISE V.

	L'.	L''.	λ'.	λ''.	C.	D.
1.	38° 14′ N.	39° 51′ N.	2° 7′ E.	4° 18′ E.	Required.	Required.
2.	49° 53′ N.	48° 28′ N.	6° 19′ W.	5° 3′ W.	Required.	Required.
3.	64° 30′ N.	60° 40′ N.	4° 20′ W.	0° 10′ E.	Required.	Required.
4.	54° 54′ S.	34° 22′ S.	60° 28′ W.	18° 24 W.	Required.	Required.
5.	17° 0′ N.	20° 0′ N.	180° 0′ E.	177° 0′ E.	Required.	Required.
6.	45° 15′ N.	Required.	35° 26′ W.	Required.	N. 49° E.	175
7.	55° 1′ N.	Required.	1° 25′ E.	Required.	N. 10° E.	246
8.	50° 48′ N.	Required.	9° 10′ W.	Required.	S. 41° W.	275
9.	37° 0′ N.	51° 18′ N.	48° 20′ W.	Required.	Required.	1027
10.	51° 15′ N.	37° 5′ N.	9° 50′ W.	Required.	S.W. by S.	Required.

11. Required the course and distance from Toulon to Valencia, by Mercator's sailing:

Toulon $\begin{cases} L = 43° \ 8' \text{ N.} \\ \lambda = 5° \ 56' \text{ E.} \end{cases}$ Valencia $\begin{cases} L = 39° \ 27' \text{ N.} \\ \lambda = 0° \ 19' \text{ W.} \end{cases}$

12. Required the compass course and distance from Cape East, New Zealand, to San Francisco; variation 14° 20′ E., and deviation 5° 40′ E.:

Cape East $\begin{cases} L = 37° 40' \text{S.} \\ \lambda = 178° 36' \text{E.} \end{cases}$ San Francisco $\begin{cases} L = 37° 48' \text{N.} \\ \lambda = 122° 24' \text{W.} \end{cases}$

13. Required the course and distance from Cape Lopatka to Callao:

Cape Lopatka $\begin{cases} L = 51° \ 2' \text{N.} \\ \lambda = 156° \ 50' \text{E.} \end{cases}$ Callao $\begin{cases} L = 12° \ 4' \text{S.} \\ \lambda = 77° \ 14' \text{W.} \end{cases}$

14. A ship from latitude 20° 40′ N. sails N.E. by N. until she is in latitude 27° 16′ N. Required the distance and difference of longitude.

15. A ship from Cape Clear, in latitude 51° 26′ N. and longitude 9° 29′ W., sails S.W. by S. until the distance run is 1022 miles. Find the latitude and longitude in by Mercator's and Middle Latitude Sailings. Which method is preferable?

§ 17. TRAVERSE SAILING.

When a ship reaches her destination by sailing on different courses, the difference of latitude may be found by computing the change in latitude for each course separately, as in Plane Sailing, and taking the algebraic sum; or, more conveniently (but less accurately), by writing the north latitudes in one column and the south latitudes in another, finding the sum of each column and taking the arithmetical difference, giving it the name (N. or S.) of the greater sum. In like manner, the total departure may be found approximately.

Evidently the Traverse Table may be used, and the work arranged as in Rectangular Surveying.

When the difference of latitude and departure have been found, the difference of longitude may be determined by Middle Latitude or Mercator's Sailings.

EXAMPLE 1. A ship in latitude 40° N. and longitude 67° 4′ W., sails N.W. 60 miles, then N. by W. 52 miles, then W.S.W. 83 miles. Find the latitude and longitude in.

C.		D.	N.	S.	E.	W.
N.W.	4 pts.	60	42.4			42.4
N. by W.	1 "	52	51.0			10.1
W.S.W.	6 "	83		31.8		76.7
Hence, L_d = 61.6 miles N.			93.4	31.8	0	129.2
= 1° 2′ N.			31.8			0.
p = 129.2 W.			61.6			129.2

L''.

$L' = 40° 0' \text{ N.}$

$L_d = 1° 2' \text{ N.}$

$L'' = 41° 2' \text{ N.}$

L_m.

$L' = 40° 0' \text{ N.}$

$L'' = 41° 2' \text{ N.}$

$2\overline{)81° 2'}$

$L_m = 40° 31'$

$90°$

$\text{co } L_m = 49° 29'$

λ''.

$\lambda' = 67° 4' \text{ W.}$

$\lambda_d = 2° 51' \text{ W.}$

$\lambda'' = 69° 55'$

The difference of longitude may be found by computation, using formula [5], or by inspection, as in § 15, Ex. 2. 49° is found at the bottom of page 118, Table IX., and opposite the departure 129.1 is distance 171, which is the difference of longitude required. 171′ = 2° 51′.

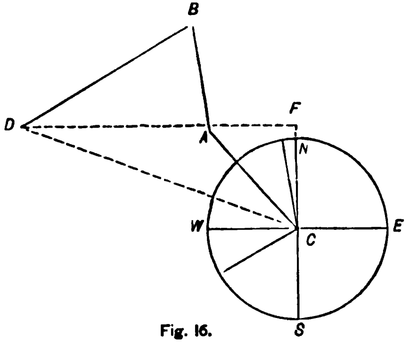


Fig. 16.

To Plot the Courses.

Let C (Fig. 16) be the centre of a circle representing the card of the compass. Draw CA, N.W. from C, equal to 60, to any convenient scale. Make an edge of a pair of parallel rulers coincide with the N. by W. rhumb of the compass diagram, extend until the other edge embraces A, then draw AB equal to 52. Make an edge of the ruler coincide with the W.S.W.

rhumb of the compass diagram, and extend until the other edge embraces B, and draw BD equal to 83. D will represent the position of the vessel.

Upon CN produced let fall the perpendicular DF. Then CF will represent the total difference of latitude, CD the distance made good, and DCF the course made good.

The departure may be regarded as measured on the parallel $\frac{1}{2}(L' + L'')$.

Example 2. The U.S. Steamer *Swatara* left latitude 37° 25′ N. and longitude 3° 14′ E., at noon, Feb. 25, 1880, and during the next 24 hours sailed on the (corrected) courses in the following tablet. Find the latitude and longitude in by account, at noon, Feb. 26.

C.		D.	N.	S.	E.	W.
N.E. by E. ¾ E.	5¾ pts.	2.0	0.9		1.8	
E. by N.	7 "	28.0	5.5		27.5	
E. ½ S.	7½ "	3.0		0.3	3.0	
E.S.E. ½ E.	6½ "	2.8		0.8	2.7	
S.S.E. ½ E.	2½ "	3.4		3.0	1.6	
N. by W. ¼ W.	1¼ "	4.2	4.1			1.0
N. ¼ E.	¼ "	4.4	4.4		0.2	
N. by E. ¼ E.	1¼ "	10.6	10.3		2.6	
N. ½ E.	½ "	11.4	11.4		1.1	
N. ½ W.	½ "	18.6	18.5			1.8
N. by W.	1 "	34.8	34.1			6.8
S. by E. ½ E.	1½ "	5.0		4.8	1.5	
Hence, $L_d = 80.3 = 1^\circ\ 20'$ N., and $p = 32.4$ E.			89.2 8.9 80.3	8.9	42.0 9.6 32.4	9.6

L''.

$L' = 37^\circ\ 25'$ N.

$L_d = \ \ 1^\circ\ 20'$ N.

$L'' = 38^\circ\ 45'$ N.

L_m.

$L' = 37^\circ\ 25'$ N.

$L'' = 38^\circ\ 45'$ N.

$2)\underline{76^\circ\ 10'}$

$L_m = 38^\circ\ \ 5'$

$\text{co}\,L_m = 51^\circ\ 55'$

λ''.

$\lambda' = 3^\circ\ 14'$ E.

$\lambda_d = 0^\circ\ 41'$ E.

$\lambda'' = 3^\circ\ 55'$ E.

EXERCISE VI.

Required the difference of latitude and the departure:

1.

C.	D.
S.S.W.	48
S.W. by S.	36
N.E.	24

2.

C.	D.
S. ½ E.	18
S.W. ½ S.	37
S.S.W. ¼ W.	56

3.

C.	D.
S.S.W. ¼ W.	43
S.S.W. ½ W.	39
S. by W. ½ W.	27

4.

C.	D.
N. 25° W.	16.4
N. 8° E.	7.8
N. 19° E.	13.7
N. 76° E.	39.6

5.

C.	D.
W.N.W. ¼ W.	21
N.N.E. ¾ E.	9
N. by E. ¾ E.	9
S.S.W. ¼ W.	30

6.

C.	D
S. 83° W.	23.0
S. 48° E.	25.2
N. 48° W.	27.1
N. 36° W.	21.0

7.

C.	D.
S. 17° E.	48
S. 45° W.	19
N. 36° W.	18
N. 41° W.	50
E.	36

8.

C.	D.
N.N.E.	31
E.N.E.	35
E. by S.	36
S.S.E.	51
S. by E.	60

9.

C.	D.
S. 44° E.	69
S. 85° E.	68
S. 27° E.	25
N. 37° W.	5
N. 20° W.	13

§ 18. THE DAY'S WORK.

When a ship is starting out upon a voyage, just before she passes out of sight of land, the proper officer notes the bearing, and estimates the distance from the ship, of some object on shore whose latitude and longitude are known. This operation is called **Taking the Departure.** After this has been done, a record is kept on the *log-board* of the courses and distances run. From the log-board the record is transferred to the *log-book*, in which it is arranged as in the following tablets. The column headed "H" contains the hours from noon to noon. The column headed "K" contains the knots per hour the ship has sailed; and the next column contains the tenths of knots. The remainder of the tablet does not require explanation.

From the record of the log-book the position of the ship is computed by Traverse Sailing each day at noon, or at any other hour when the position is desired, in the manner described in the following examples.

In computing the day's work, the first course of the traverse is found by reversing the departure course. For, if, as in Example 1, the bearing of the object *from the ship* be N.E. ¼ E. 15 miles, then it follows that the bearing of the ship *from the object on shore* is S.W. ¼ W. 15 miles.

This method of finding the place of a ship is subject to many errors. It is impossible to determine the course accurately: the deviation is almost constantly changing; the effect of tides and currents can be estimated only approximately; the log gives the distance roughly; the speed varies during the hour; etc., etc.

Hence, whenever it is possible to do so, the results obtained by dead-reckoning should be verified by celestial observation. (See Chapter III.) The results obtained by the latter method should be given the preference.

A number of the following exercises are taken from recent examination papers of the British Marine Board; others are actual transcripts from the log-books of vessels.

The log-book used in the U. S. Navy contains, in addition to the data of the following tablets, columns in which are recorded the force of the wind, the height of the barometer, the temperature of the air by dry and wet bulb thermometers, the surface temperature of the water, the state of the weather, the forms of clouds, the proportion of clear sky, the state of the sea, a record of the sail the vessel is under at the end of the watch; also, a record of the miscellaneous events of the day.

EXAMPLE 1.

H.	*Courses.*	*K.*	*⅒*	*Winds.*	*Leeway.*	*Deviation.*	*Remarks, etc.*
1	W.S.W.	10	8	N.W.	½ pt.	11° W.	The departure was taken as follows: A point of land in latitude 37° 3′ N., longitude 9° 0′ W., bearing by compass N.E. ¼ E., dist. 15 miles. Ship's head W.S.W. Deviation as per log.
2		11	4				
3		11	4				
4		11	4				
5	N.W. ½ N.	12	2	W.S.W.	¼ pt.	17° W.	
6		12	3				
7		12	3				
8		12	2				
9	N.N.W.	9	6	W.	¾ pt.	11° W.	
10		9	5				
11		9	5				
12		9	4				Variation 22° 30′ W.
1	N.W. by W.	7	8	S.W. by W.	1¼ pts.	20° W.	
2		7	6				
3		7	4				A current set the ship S. W. by W. ¾ W. (correct magnetic) 8 miles, from the time the departure was taken to the end of the day.
4		8	2				
5	S.W. ½ S.	9	3	S.S.E.	1 pt.	6° W.	
6		8	7				
7		9	3				
8		8	7				
9	W. ½ S.	10	3	S. by W.	½ pt.	13° W.	
10		10	2				
11		10	2				
12		10	3				

The departure course.

The opposite point to N.E. ¼ E. is S.W. ¼ W. The ship's head being W. S. W., the deviation is the same as for the first course, viz., 11° W.

S.W. ¼ W. = 4¼ pts. R. of S.
or 47° 49′ R. of S.
Dev. 11° 0′ W. } = 33° 30′ L.
Var. 22° 30′ W. }
True course = 14° 19′ R. of S.

Hence, the first course and distance of the traverse are

S. 14° W. 15 miles.

First course.

W. S. W. = 6 pts. R. of S.
Leeway = ½ " L.
5½ pts. R. of S.
or 61° 53′ R. of S.
Dev. 11° 0′ W. } = 33° 30′ L.
Var. 22° 30′ W. }
True course = 28° 23′ R. of S.

The distance (45 miles) for this course is found by adding together the distances sailed each hour up to 5 o'clock, when the course was changed.

Hence, the second course and distance of the traverse are

S. 28° W. 45 miles.

Second course.

N.W. ½ N. = 3½ pts. L. of N.
Leeway = ¼ " R.

3¼ pts. L. of N.
or 36° 34′ L. of N.

$$\left.\begin{array}{l}\text{Dev. } 17^\circ\ 0'\text{ W.}\\ \text{Var. } 22^\circ\ 30'\text{ W.}\end{array}\right\} = 39^\circ\ 30'\text{ L.}$$

True course = 76° 4′ L. of N.

The distance (49 mi.) is obtained by adding together the hourly distances from 5 to 9 o'clock.

Hence, the third course and distance of the traverse are

N. 76° W. 49 miles.

Third course.

N.N.W. = 2 pts. L. of N.
Leeway = ¾ " R.

1¼ pts. L. of N.
or 14° 4′ L. of N.

$$\left.\begin{array}{l}\text{Dev. } 11^\circ\ 0'\text{ W.}\\ \text{Var. } 22^\circ\ 30'\text{ W.}\end{array}\right\} = 33^\circ\ 30'\text{ L.}$$

True course = 47° 34′ L. of N.

The distance (38 mi.) is obtained by adding together the hourly distances from 9 to 1 o'clock.

Hence, the fourth course and distance of the traverse are

N. 48° W. 38 miles.

Fourth course.

N.W. by W. = 5 pts. L. of N.
Leeway = 1¼ " R.

3¾ pts. L. of N.
or 42° 11′ L. of N.

$$\left.\begin{array}{l}\text{Dev. } 20^\circ\ 0'\text{ W.}\\ \text{Var. } 22^\circ\ 30'\text{ W.}\end{array}\right\} = 42^\circ\ 30'\text{ L.}$$

True course = 84° 41′ L. of N.

Hence, the fifth course and distance of the traverse are

N. 85° W. 31 miles.

Fifth course.

S.W. ½ S. = 3½ pts. R. of S.
Leeway = 1 " R.

4½ pts. R. of S.
or 50° 38′ R. of S.

$$\left.\begin{array}{l}\text{Dev. } 6^\circ\ 0'\text{ W.}\\ \text{Var. } 22^\circ\ 30'\text{ W.}\end{array}\right\} = 28^\circ\ 30'\text{ L.}$$

True course = 22° 8′ R. of S.

Hence, the sixth course and distance of the traverse are

S. 22° W. 36 miles.

Sixth course.

W. ½ S. = 7½ pts. R. of S.
Leeway = ½ " R.

8 pts. R. of S.
or 90° 0′ R. of S.

$$\left.\begin{array}{l}\text{Dev. } 13^\circ\ 0'\text{ W.}\\ \text{Var. } 22^\circ\ 30'\text{ W.}\end{array}\right\} = 35^\circ\ 30'\text{ L.}$$

True course = 54° 30′ R. of S.

Hence, the seventh course and distance of the traverse are

S. 55° W. 41 miles.

Current course.

S.W. by W. ¾ W. = 5¾ pts. R. of S.
or 64° 41′ R. of S.
Variation = 22° 30′ L.

True course = 42° 11′ R. of S.

Hence, the eighth course and distance of the traverse are

S. 42° W. 8 miles.

The Traverse.

	Courses.	Dist.	N.	S.	E.	W.
1	S. 14° W.	15		14.6		3.6
2	S. 28° W.	45		39.7		21.1
3	N. 76° W.	49	11.9			47.5
4	N. 48° W.	38	25.4			28.2
5	N. 85° W.	31	2.7			30.9
6	S. 22° W.	36		33.4		13.5
7	S. 55° W.	41		23.5		33.6
8	S. 42° W.	8		5.9		5.4
$L_d = 77.1$ miles			40.0	117.1		183.8
$= 1° 17'$ S.,				40.0		
$p = 183.8$ W.				77.1		

To find the latitude in:

$L' = 37° \ 3'$ N.
$L_d = \ 1° 17'$ S.
$L'' = 35° 46'$ N.

Also, $L_m = 36° 25'$.

To find the direct course:

$$\tan C = \frac{p}{L_d}. \qquad [3]$$

$\log p = 2.26435$
$\text{colog } L_d = 8.11295$
$\log \tan C = 10.37730$
$\therefore C = $ S. 67° W

To find the direct distance:

$$D = \frac{p}{\sin C}. \qquad [1]$$

$\log p = 2.26435$
$\text{colog} \sin C = 0.03517$
$\log D = 2.29952$
$\therefore D = 199$ miles.

To find the difference of longitude:

$$\lambda_d = p \sec L_m. \qquad [4]$$

$\log p = 2.26435$
$\log \sec L_m = 0.09435$
$\log \lambda_d = 2.35870$
$\therefore \lambda_d = 228' = 3° 48'$ W.

To find the longitude in:

$\lambda' = \ 9° \ 0'$ W.
$\lambda_d = \ 3° 48'$ W.
$\lambda'' = 12° 48'$ W.

EXAMPLE 2.

H.	*Courses.*	*K.*	$\frac{1}{10}$	*Winds.*	*Leeway.*	*Deviation.*	*Remarks.*
1	S.W.	6	0	W.N.W.	1 pt.	0	Latitude from 28° 30′ S. Longitude from 10° 15′ E.
2		6	3				
3		6	4				
4		6	0				
5		5	3				
6	S. by W.	6	0	W. by S.	1 pt.		
7		5	1				
8		5	4				
9		5	2				
10		5	3				Variation 1½ pts. W.
11		5	0				
12	S.S.W.	5	2	W.	1 pt.		
1		5	2				
2		5	0				
3		4	6				A current set S. (correct magnetic) 2½ miles per hour, from noon to noon.
4		5	0				
5		5	0				
6	S.E. by S.	5	2	S.W. by S.	1 pt.		
7		5	4				
8		5	5				
9		6	0				
10		6	0				
11		5	4				
12		5	5				

First course.

S.W. = 4 pts. R. of S.
Leeway = 1 " L.
3 pts. R. of S.
Variation = 1½ " L.
True course = 1½ pts. R. of S.
= S. 1½ pts. W.

Second course.

S. by W. = 1 pt. R. of S.
Leeway = 1 " L.
0 pts. R. of S.
Variation = 1½ " L.
True course = 1½ " L. of S.
= S. 1½ pts. E.

Third course.

S.S.W. = 2 pts. R. of S.
Leeway = 1 " L.
1 pt. R. of S.
Variation = 1½ " L.
True course = ½ pt. L. of S.
= S. ½ pt. E.

Fourth course.

S.E. by S. = 3 pts. L. of S.
Leeway = 1 " L.
4 pts. L. of S.
Variation = 1½ " L.
True course = 5½ pts. L. of S.
= S. 5½ pts. E.

The current course.

S. = 0 pts. L. of S.
Variation = 1½ " L.

True course = 1½ pts. L. of S.
= S. 1½ pts. E.

The Traverse.

	Courses.	Dist.	N.	S.	E.	W.
1	S. 1½ pts. W.	30		28.7		8.7
2	S. 1½ pts. E.	32		30.6	9.3	
3	S. ½ pt. E.	30		29.9	2.9	
4	S. 5½ pts. E.	39		18.4	34.4	
5	S. 1½ pts. E.	60		57.4	17.4	
L_d = 165 miles = 2° 45′ S.				165.0	64.0	8.7
L' = 28° 30′ S.					8.7	
L'' = 31° 15′ S.					55.3	

L' = 28° 30′ S. Mer. Parts = 1774.3
L'' = 31° 15′ S. Mer. Parts = 1963.6

Mer. L_d = 189.3

To find the difference of longitude:

$$\lambda_d = \frac{\text{Mer. } L_d \times p}{L_d}. \qquad [9]$$

Mer. L_d = 189.3, log = 2.27715
p = 55.3, log = 1.74273
L_d = 165.0, colog = 7.78252

log λ_d = 1.80240

$\therefore \lambda_d$ = 63.4 = 1° 3′ E.

λ' = 10° 15′ E.
λ_d = 1° 3′ E.

λ'' = 11° 18′ E.

The distance made good may be found by [2], and the course made good by [3] or [8].

Exercise VII.

Find the latitude and longitude in from the following log-book records:

1.

H.	*Courses.*	*K.*	*T.*	*Winds.*	*Leeway.*	*Deviation.*	*Remarks.*
1	N.N.E.	6	3	W.	0	0	**Latitude from 46° 28′ N.** **Longitude from 22° 18′ W.**
2		6	2				
3		6	5				
4		6	4				
5		6	0				
6	E.N.E.	6	1	N.W.			
7		6	6				
8		5	8				
9		5	6				
10		5	4				
11		5	5				
12	E. by S.	5	3	N.			**Variation 0.**
1		5	9				
2		6	2				
3		6	0				
4		6	3				
5		6	4				
6	S.S.E.	7	0	N.			**The tide set S. by E., 2½ mi. per hour, during the 24 hours.**
7		6	8				
8		7	3				
9		7	5				
10		7	1				
11		7	9				
12		7	3				

2.

H.	*Courses.*	*K.*	*T.*	*Winds.*	*Leeway.*	*Deviation.*	*Remarks.*
1	S. by W.	6	2	W.	0	0	Latitude from 33° 40′ N. Longitude from 16° 20′ W.
2		6	0				
3		6	3				
4		7	0				
5		7	2				
6		7	3				
7	S.W. by S.	7	2	W. by N.	1 pt.		
8		7	2				
9		7	4				
10		7	6				
11		7	4				
12		8	1				Variation 12° 20′ W.
1		8	0				
2		8	5				
3		8	2				
4	S.W. by W.	7	5	N.W.	0		
5		7	3				
6		6	6				A current set W. S.W. (correct magnetic), 1½ miles per hour from noon to noon.
7		6	4				
8		6	0				
9		6	2				
10		6	1				
11		6	3				
12		6	1				

3.

H.	*Courses.*	*K.*	*T.*	*Winds.*	*Leeway.*	*Deviation.*	*Remarks.*
1	N. by E.	6	7	E. by N.	1 pt.	0	Latitude from 19° 30′ S. Longitude from 0° 10′ E.
2		6	2				
3		6	4				
4		6	3				
5		6	1				
6		6	0				
7	N.	5	8	E.N.E.	1 pt.		
8		5	4				
9		5	0				
10		5	3				
11		5	6				
12		5	9				Variation 13° 30′ W.
1		5	7				
2	N.N.W.	6	4	N.E.	1 pt.		
3		6	8				
4		7	0				
5		7	3				
6		7	6				The current set W. N. W. (correct magnetic), ½ mile per hour, from noon to noon.
7		7	5				
8		7	0				
9		7	2				
10		7	4				
11		6	3				
12		6	0				

4.

H.	*Courses.*	*K.*	*T.*	*Winds.*	*Leeway.*	*Deviation.*	*Remarks.*
1	S.E. by E.	13	2	N.	0	11° E.	**A point of land, latitude 47° 31′ N., longitude 52° 33′ W., bearing by compass W.S.W., distant 18 miles. Ship's head S.E. by E. Deviation as per log.**
2		13	3				
3		12	5				
4		13					
5	S.E.	11		E.N.E.	½ pt.	9° E.	
6		10	5				
7		10	4				
8		11	1				
9	E. by N.	8	2	S.E. by S.	1 pt.	17° E.	
10		9	4				
11		9	4				
12		9					
1	E.N.E.	6	8	S.E.	1½ pts.	15° E.	**Variation 28° W.**
2		6	7				
3		6	5				
4		7					
5	S.S.E.	5	8	E.	2 pts.	7° E.	**A current set the ship (correct magnetic) S. by E. 12 miles from the time the departure was taken to the end of the day.**
6		5	8				
7		6	4				
8		6					
9	S.E. by S.	7		E. by N.	1½ pts.	8° E.	
10		7	3				
11		7	4				
12		7	3				

5.

H.	*Courses.*	*K.*	*T.*	*Winds.*	*Leeway.*	*Deviation.*	*Remarks.*
1	S.S.E.	5	6	E.	2¼ pts.	3° E.	**A point of land in latitude 62° N., longitude 150° E., bearing by compass W. by S. ¼ S., distant 17 mi. Ship's head S.S.E.; deviation as per log.**
2		5	6				
3		5	0				
4		4	8				
5	S.S.W. ½ W.	4	7	W.	2¾ pts.	4° W.	
6		4	8				
7		5	2				
8		5	3				
9	W.S.W.	5	0	S.	2½ pts.	9° W.	
10		6	0				
11		6	5				
12		6	5				
1	W. ½ N.	6	6	N. by E.	0	11° W.	**Variation 31° E.**
2		7	0				
3		6	4				
4		6	0				
5	E.	4	6	S.S.E.	2½ pts.	10° E.	
6		5	0				**A current set the ship (correct magnetic) N.N.E. 21 mi. from the time the departure was taken to the end of the day.**
7		4	8				
8		4	6				
9	E.S.E.	4	0	S. by W.	0	9° E.	
10		4	5				
11		4	5				
12		5	0				

6.

H.	*Courses.*	*K.*	*T.*	*Winds.*	*Leeway.*	*Deviation.*	*Remarks.*
1	S.W. ½ W.	14	5	S.E.	0	6° W.	**A point, latitude 50° 12′ S., longitude 179° 40′ W., bearing by compass N.¾W., distant 19 miles.** **Ship's head S.W. ½ W. Deviation as per log.**
2		14	2				
3		14	6				
4		14	7				
5	N. ¾ E.	4	0	E.N.E.	3¼ pts.	3° E.	
6		3	6				
7		3	6				
8		3	8				
9	S. by E. ½ E.	2	4	S.W. ½ W.	2¾ pts.	6° E.	
10		2	3				
11		2	3				
12		2	0				
1	W. by S.	12	2	S. by W.	¼ pt.	14° W.	**Variation 14° E.**
2		12	4				
3		12	6				
4		12	8				
5	E.N.E.	3	0	S.E.	2½ pts.	19° E.	**A current set the ship (correct magnetic) S.W. ¼ W. 42 miles from the time the departure was taken to the end of the day.**
6		2	3				
7		3	4				
8		3	3				
9	S.S.W. ½ W.	5	6	S.E.	1¾ pts.	4° W.	
10		5	7				
11		5	3				
12		5	4				

7.

H.	*Courses.*	*K.*	*T.*	*Winds.*	*Leeway.*	*Deviation.*	*Remarks.*
1							**Left Malaga at 8 P.M.** } L 36° 42′ N. λ 4° 25′ W.
2							
3							
4							
5							
6							
7							
8	S.E.	1	5		0	¼ pt. E.	
9		6	4				
10		6	6				
11		6	8				
12		6	5				
1	E.S.E. ¼ E.	7	2		0	0	**Variation 1¾ pts. W.**
2		7	2				
3		8	2				
4		8	6				
5		8	0				
6		7	6				
7		7	6				
8		7	6				
9		7	5				
10		5	7				
11	E.	4	2		0	½ pt. E.	
12		4	5				

§ 19. GREAT CIRCLE SAILING.

It may be proven that the shortest distance between two places on the surface of the earth is the less of the two arcs of a great circle lying between these places. The rhumb-line is the arc of a great circle only when it coincides with the equator or a meridian.

The advantages of great circle sailing are:

1. The distance is less than by any other route.
2. The great circle track gives the real direction of the desired port, and this frequently determines which tack the ship should be put on.

To sail on a great circle would require a constant change of course. This difficulty is practically overcome by sailing from point to point of the great circle on rhumb-lines.

Definition. The *vertex* of a great circle track is the point nearest the pole. Hence, the vertex is the foot of the perpendicular let fall upon the great circle from the pole.

The best practical method of finding the courses, distance, etc., of the great circle track is by the use of Godfray's Course and Distance Diagram, which affords a simple graphic solution. The solution by Spherical Trigonometry is given here.

Let CC' (Fig. 17) represent the great circle track between C and C', P the pole, and PC and PC' meridians.

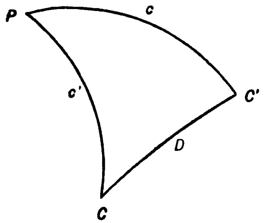


Fig. 17.

The angle $C =$ the first course from C.

$C' =$ the first course from C'.

$P = \lambda_d =$ the difference of longitude of C and C'.

$PC = c' =$ the polar distance of $C = 90° \mp$ latitude of C.

$PC' = c\ =$ the polar distance of $C' = 90° \mp$ latitude of C'.

$CC' = D =$ the length of the great circle track.

To find the first course.

In the triangle *CPC'* (Fig. 17) by Napier's Analogies (Spher. Trig. § 53):

$$\left.\begin{aligned} \tan \tfrac{1}{2}(C+C') &= \frac{\cos \frac{1}{2}(c-c')}{\cos \frac{1}{2}(c+c')} \cot \tfrac{1}{2} \lambda_d, \\ \tan \tfrac{1}{2}(C-C') &= \frac{\sin \frac{1}{2}(c-c')}{\sin \frac{1}{2}(c+c')} \cot \tfrac{1}{2} \lambda_d, \\ C &= \tfrac{1}{2}(C+C') + \tfrac{1}{2}(C-C'), \\ C' &= \tfrac{1}{2}(C+C') - \tfrac{1}{2}(C-C'). \end{aligned}\right\} \qquad [10]$$

To find the distance.

By Gauss's Equations (Spher. Trig. § 53):

$$\cos \tfrac{1}{2} D = \frac{\cos \frac{1}{2}(c+c')}{\cos \frac{1}{2}(C+C')} \sin \tfrac{1}{2} \lambda_d. \qquad [11]$$

See also Spher. Trig. § 62.

To find the latitude of the vertex.

Let *PV* (Fig. 18) be the perpendicular from *P* upon the arc of a great circle *CC'*. *V* is the vertex of *CC'*.

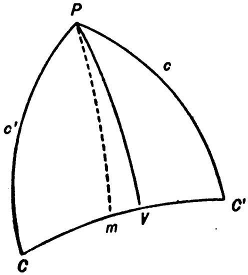


Fig. 18.

In the right triangle *CPV*,

$$\left.\begin{aligned} \sin PV &= \sin c' \sin C, \\ \text{Latitude of } V &= 90^\circ - PV. \end{aligned}\right\} \qquad [12]$$

See Spher. Trig. § 47.

To find the longitude of the vertex from C.

In the right triangle *CPV*,

$$\cos c' = \cot C \cot CPV.$$

Hence,

$$\cot CPV = \cos c' \tan C. \qquad [13]$$

Combining this result with the longitude of *C*, the longitude of *V* is found.

To find the latitude of intermediate points of given longitude.

Let m (Fig. 18) be a point whose longitude from V is assumed.

In the right triangle PmV,

$$\cos mPV = \tan PV \cot Pm.$$

Hence,
$$\left.\begin{aligned} \cot Pm &= \cos mPV \cot PV, \\ \text{Latitude of } m &= \pm (90° - Pm). \end{aligned}\right\} \qquad [14]$$

Points m, etc., are usually assumed on each side of V, differing from V and from each other 5° or 10° in longitude. In this manner, the latitude and longitude of as many points as we please between C and C' may be determined. The course and distance from point to point by rhumb-line may then be found by Middle Latitude or Mercator's Sailings.

When either of the angles C or C' is greater than 90°, and the other is less than 90°, the vertex will be on CC' produced through the vertex of the angle which exceeds 90°. (See Fig. 20.)

NOTE. If L' and L'' represent the latitudes of C and C' respectively, the first formula of set [10] becomes

$$\tan \tfrac{1}{2}(C + C') = \frac{\cos \frac{1}{2}(L' - L'')}{\sin \frac{1}{2}(L' + L'')} \cot \tfrac{1}{2} \lambda_a,$$

provided both places are on the same side of the equator. The student will find it a valuable exercise to express all of the preceding formulas of this section similarly.

To plot the Great Circle Track on Mercator's Chart.

1. By means of great circle charts. On these charts the great circle is represented by a straight line; hence, the latitude of each intersection with the meridians is known. The corresponding points may be marked on the meridians on Mercator's chart and a line traced through these points. This curved line will represent the great circle track.

2. By computing the latitudes corresponding to certain longitudes (or *vice versa*) of the great circle, marking these points on Mercator's chart, and tracing the curve through them.

EXAMPLE 1. Required the initial courses, the distance, and the latitude and longitude of the vertex of the great circle track joining C and C'. Given:

Lat. of $C = 54° 18'$ N.
Lat. of $C' = 7° 12'$ S.
Long. of $C = 142° 38'$ E.
Long. of $C' = 90° \ 0'$ W.

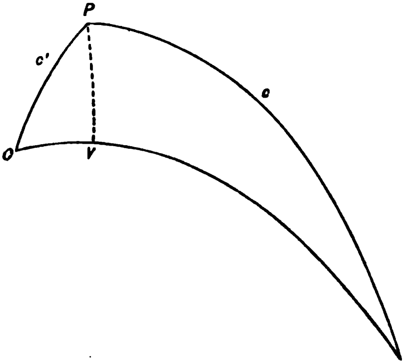


Fig. 19.

Referring to the triangle CPC' (Fig. 19),

$c = 90° + 7° 12' = 97° 12'$,
$c' = 90° - 54° 18' = 35° 42'$,
$\lambda_d = P = (180° - 142° 38') + (180° - 90°) = 127° 22'$.

To find the initial courses:

$$\left.\begin{aligned}\tan \tfrac{1}{2}(C + C') &= \frac{\cos \frac{1}{2}(c - c')}{\cos \frac{1}{2}(c + c')} \cot \tfrac{1}{2}\lambda_d,\\ \tan \tfrac{1}{2}(C - C') &= \frac{\sin \frac{1}{2}(c - c')}{\sin \frac{1}{2}(c + c')} \cot \tfrac{1}{2}\lambda_d.\end{aligned}\right\} \qquad [10]$$

$\frac{1}{2}(c - c') = 30° 45'$	log cos = 9.93420	log sin = 9.70867
$\frac{1}{2}(c + c') = 66° 27'$	colog cos = 0.39843	colog sin = 0.03777
$\frac{1}{2}\lambda_d = 63° 41'$	log cot = 9.69425	log cot = 9.69425
$\frac{1}{2}(C + C') = 46° 46'$,	log tan = 10.02688	
$\frac{1}{2}(C - C') = 15° 25'$,		log tan = 9.44069

∴ $C = 62° 11'$, or N. 62° 11′ E.
and $C' = 31° 21'$, or N. 31° 21′ W.

To find the distance:

$$\cos \tfrac{1}{2} D = \frac{\cos \frac{1}{2}(c + c')}{\cos \frac{1}{2}(C + C')} \sin \tfrac{1}{2}\lambda_d. \qquad [11]$$

$\frac{1}{2}(c + c') = 66° 27'$	log cos = 9.60157
$\frac{1}{2}(C + C') = 46° 46'$	colog cos = 0.16433
$\frac{1}{2}\lambda_d = 63° 41'$	log sin = 9.95248
$\frac{1}{2}D = 58° 29'$,	log cos = 9.71838

∴ $D = 116° 58'$
= 7018 miles.

To find L of the vertex:

$$\sin PV = \sin c' \sin C. \qquad [12]$$

$c' = 35^\circ 42'$, log sin = 9.76607
$C = 62^\circ 11'$, log sin = 9.94667
$PV = 31^\circ\ 4'$, log sin = 9.71274
L of $V = 90^\circ - 31^\circ 4' = 58^\circ 56'$ N.

To find λ of the vertex:

$$\cot CPV = \cos c' \tan C. \qquad [13]$$

log cos c' = 9.90960
log tan C = 10.27769
log cot CPV = 10.18729
$\therefore CPV = 33^\circ\ 1'$
λ of $C = 142^\circ 38'$ E.
λ of $V = 175^\circ 39'$ E.

Example 2. The Cunard steamers in sailing from New York to Liverpool keep approximately on the great circle between a point off the American coast, in latitude 42° N., longitude 50° W., and Fastnet Rock off the Irish coast, in latitude 51° 24′ N., longitude 9° 36′ W. Required the initial course, the distance between these points, and the latitude and longitude of the vertex.

In the triangle CPC' (Fig. 20) let C represent the initial point, C' Fastnet, and P the pole.

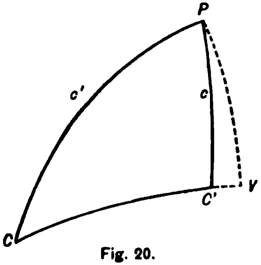


Fig. 20.

$c' = 90^\circ - 42^\circ\ 0' = 48^\circ\ 0'$
= colatitude of C.
$c = 90^\circ - 51^\circ 24' = 38^\circ 36'$
= colatitude of C'.
$\lambda_d = P = 50^\circ - 9^\circ 36' = 40^\circ 24'$
= difference of longitude of C and C'.

To find the initial course:

$$\left.\begin{aligned} \tan \tfrac{1}{2}(C' + C) &= \frac{\cos \frac{1}{2}(c' - c)}{\cos \frac{1}{2}(c' + c)} \cot \tfrac{1}{2}\lambda_d, \\ \tan \tfrac{1}{2}(C' - C) &= \frac{\sin \frac{1}{2}(c' - c)}{\sin \frac{1}{2}(c' + c)} \cot \tfrac{1}{2}\lambda_d. \end{aligned}\right\} \qquad [10]$$

$\frac{1}{2}(c' - c) = 4^\circ 42'$	log cos = 9.99854	log sin = 8.91349
$\frac{1}{2}(c' + c) = 43^\circ 18'$	colog cos = 0.13800	colog sin = 0.16379
$\frac{1}{2}\lambda_d = 20^\circ 12'$	log cot = 10.43424	log cot = 10.43424
$\frac{1}{2}(C' + C) = 74^\circ 58'$,	log tan = 10.57078	
$\frac{1}{2}(C' - C) = 17^\circ 59'$,		log tan = 9.51152

$\therefore C = 56^\circ 59'$, or N. 56° 59′ E.

To find the distance:

$$\cos \tfrac{1}{2} D = \frac{\cos \tfrac{1}{2}(c + c')}{\cos \tfrac{1}{2}(C + C')} \sin \tfrac{1}{2} \lambda_d. \qquad [11]$$

$\tfrac{1}{2}(c + c') = 43° 18'$	$\log \cos = 9.86200$
$\tfrac{1}{2}(C + C') = 74° 58'$	$\operatorname{colog} \cos = 0.58606$
$\tfrac{1}{2}\lambda_d = 20° 12'$	$\log \sin = 9.53819$
$\tfrac{1}{2} D = 14° 21'$,	$\log \cos = 9.98625$
$\therefore D = 28° 42'$	
$= 1722$ miles.	

To find L of the vertex:

$$\sin PV = \sin c' \sin C. \qquad [12]$$

$c' = 48° \; 0'$,	$\log \sin = 9.87107$
$C = 56° 59'$,	$\log \sin = 9.92351$
$PV = 38° 33'$,	$\log \sin = 9.79458$

L of $V = 90° - 38° 33' = 51° 27'$ N.

To find λ of the vertex:

$$\cot CPV = \cos c' \tan C. \qquad [13]$$

$\log \cos c' = 9.82551$

$\log \tan C = 10.18721$

$\log \cot CPV = 10.01272$

$\therefore CPV = 44° 10'$

λ of $C = 50° \; 0'$ W.

λ of $V = 5° 50'$ W.

NOTE. $\tfrac{1}{2}(C + C') = 74° 57' 42''$, using this value, $D = 1730$ miles. From Sandy Hook to the company's position 1088 miles, thence to Fastnet by great circle 1730 miles; total, 2818 miles.

EXERCISE VIII.

1. Find the elements (initial courses, distance, and latitude and longitude of the vertex) of the great circle track between the Lizard, in latitude 49° 58′ N., longitude 5° 12′ W., and the Bermuda Islands, in latitude 32° 18′ N., longitude 64° 50′ W.

2. Find the elements of the great circle track between Boston (Minot's Ledge light-house), in latitude 42° 16′ N., longitude 70° 46′ W., and Cape Clear, in latitude 51° 26′ N., longitude 9° 29′ W. (Take $\tfrac{1}{2}\lambda_d = 30° 39'$.)

3. Find the elements of the great circle track between Vancouver Island, in latitude 50° N., longitude 128° W., and Honolulu, in latitude 21° 18′ N., longitude 157° 52′ W.

4. Find the elements of the great circle track between Cape Clear, in latitude 51° 26′ N., longitude 9° 29′ W., and Sandy Hook, in latitude 40° N., longitude 74° W.

5. Find the elements of the great circle track between Lizard Lights, in latitude 49° 58′ N., longitude 5° 12′ W., and Cape Frio, in latitude 23° S., longitude 42° W.

6. Find the elements of the great circle track between Cape Frio and Cape Good Hope, in latitude 34° 20′ S., longitude 18° 30′ E. (Reckon from the nearest pole.)

7. Find the elements of the great circle track between Grand Port, Mauritius, in latitude 20° 24′ S., longitude 57° 47′ E., and Perth, in latitude 32° 3′ S., longitude 115° 45′ E.

8. Find the elements of the great circle track between A, in latitude 16° 38′ N., longitude 70° 55′ W., and B, in latitude 48° 2′ N., longitude 4° 35′ W.

9. A ship sails from A, in latitude 40° S., longitude 148° 30′ E., to B, in latitude 12° 4′ S., longitude 77° 14′ W. Compare the great circle and the rhumb-line between A and B.

CHAPTER III.

NAUTICAL ASTRONOMY.*

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§ 20. THE OBSERVED ALTITUDE OF A HEAVENLY BODY.

The Sextant.

The Sextant is an instrument used by navigators for determining the angular distance between two heavenly bodies, or the angular altitude of a heavenly body above the horizon.

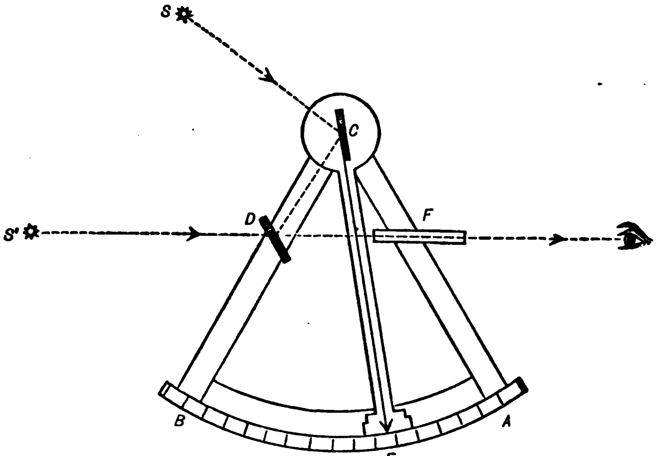

Fig. 21.

This instrument is shown in outline in Fig. 21. It consists of a frame *ABC* in the form of a sector whose arc *AB* is 60° in extent, but divided into 120 equal parts reckoned as degrees. *CE* is a movable radius which carries at *C* a mirror called the *index-glass*, whose plane is perpendicular to the plane of the sextant; and at *E* an *index* which moves over the arc *AB*. At *D* is fixed the *horizon-glass*, also perpendicular to the plane of the sextant, and parallel to *CA*; that is, parallel to the index-glass when the index is at 0 (*A*). Only a part of the horizon-glass is silvered. *F* is a telescope fixed to *CA*.

---

* For definitions see Spher. Trig., §§ 63 and 64.

To determine the angular distance between two heavenly bodies *S* and *S'*, the sextant is held in the hand so that a ray of light from *S'* passes through the unsilvered part of the horizon-glass, enters the telescope, and is seen as if by direct vision. The arm *CE* is then moved so that a ray of light from *S*, being reflected by the mirrors at *C* and *D*, enters the telescope, taking the path *SCDF*. The movement of *CE* is continued until the image of *S* appears to be in contact with *S'*, when the arc *AE* gives the angular distance required.

For, it is a principle in optics that, "*when a plane mirror is rotated in the plane of incidence, the direction of the reflected ray is changed by double the angle through which the mirror is turned.*" * Hence, the angular distance between the two bodies, which is the angle between the two lines *SC* and *S'F*, is double the angle *ACE*, or double the arc *AE*. But since half-degrees on *AB* are marked as degrees, *AE* will be the reading required.

If *S'* is the point on the sea horizon directly under *S*, the arc *AE* will give the *observed altitude* of *S*.

The **Quintant** has an arc of 72°, and is therefore capable of measuring angles up to 144°. In other respects it is like the sextant.

The **Octant** (commonly, but improperly, called the *quadrant*) has an arc of 45°, and is therefore capable of measuring angles up to 90°. In general construction it is similar to the sextant, for which it is a cheap substitute. It is much more easily handled, and yet sufficiently accurate for many purposes.*

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* Deschanel's Natural Philosophy, page 892.

In Fig. 21, the arc AB, extending from zero to the left, is called the *arc proper;* the arc to the right of zero is called the *arc of excess.* The index is *on* the arc or *off* the arc according as it is on the arc proper or on the arc of excess.

When the index-glass is parallel to the horizon-glass, the index should be at 0°. If it is not, the arc between the zero-point and the index is called the **Index Error.** The index correction is — when the index is *on* the arc, and + when the index is *off* the arc.

§ 21. CORRECTIONS OF THE OBSERVED ALTITUDE.

Refraction.

Let S (Fig. 22) represent the true position of a star, O the position of the observer on the surface of the earth, and PP' the upper limit of the atmosphere.

According to the principle of refraction, a ray of light from S meeting the atmosphere at P will be bent toward the perpendicular to the arc PP' at P. As the density of the atmosphere increases from P to O, PO will be a curve, and the star will be seen in the direction OS' tangent to PO at O.

Astronomical refraction is the difference of direction of the two lines PS and OS'.

The effect of refraction is to increase the apparent altitude of a celestial object, and is greatest when the body is near the horizon, and decreases if the altitude of the body increases.

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* The octant is fully equal to every-day work in the broad ocean; for example, during the winter months in the North Atlantic. The delicate exactness of the sextant is quite thrown away when one can get only flying shots at the horizon, from the crest of a 60-feet wave. Showers of salt spray, with the chance of an occasional knock, certainly seem less suited to the sextant than to its hardier and more humble relative. — *Captain Lecky, in "'Wrinkles' in Practical Navigation,"* p. 38.

Table XIII. gives the correction for refraction corresponding to every apparent altitude from 0° to 90°. This correction must evidently be subtracted from the apparent altitude.

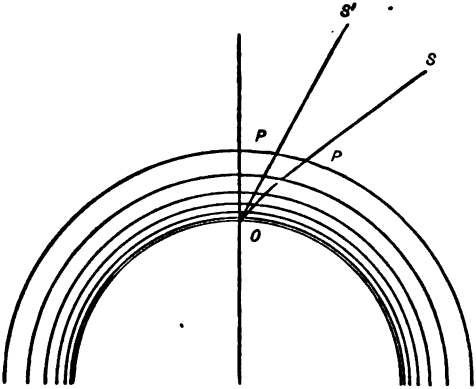

Fig. 22.

The refraction varies slightly according to the condition of the atmosphere, but this variation is not taken into account in this chapter.

Dip of the Horizon.

Let BH'' (Fig. 23) represent the level of the sea, O the position of the observer at the height OB above the sea-level, C the centre of the earth, and H'' the farthest point on the surface of the earth visible from O. On account of refraction, the visual ray $H''O$ will be a curve concave toward C, and H'' will be seen in the direction of OH' tangent to $H''O$ at O. Let OH, perpendicular to OC, represent the

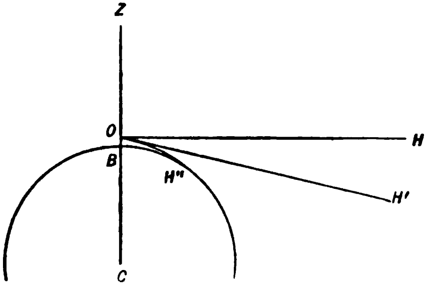

Fig. 23.

true horizon, from which the apparent altitude of the body is estimated.

The dip = angle HOH'.

The dip is very nearly proportional to the square root of the height of the observer above the sea-level. Its effect evidently is to increase the observed altitude of a heavenly body; hence, the correction for dip must be subtracted from the observed altitude.

This element is given in Table XV.

Parallax.

The difference between the positions of a heavenly body as seen from the centre of the earth and a point on its surface at the same time, is called geocentric parallax.

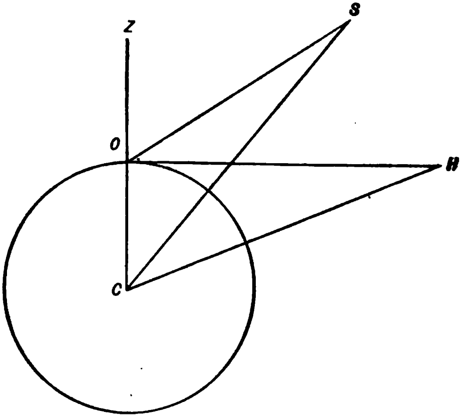

Fig. 24.

Let S (Fig. 24) represent the body, O the position of the observer, C the centre of the earth, and Z the zenith of the observer.

The geocentric parallax is the angle

$$S = ZOS - ZCS.$$

ZOS is the apparent zenith distance, and ZCS the geocentric or true zenith distance. It is evident that geocentric parallax decreases if the altitude of the body increases.

Let P = the parallax in altitude,
$Z = ZOS$ the apparent zenith distance,
$R = CO$,
$D = CS$.

In the triangle COS,

$$\frac{\sin S}{CO} = \frac{\sin COS}{CS}, \text{ or } \frac{\sin P}{R} = \frac{\sin Z}{D}.$$

Hence,
$$\sin P = \frac{R \sin Z}{D}. \qquad (a)$$

If the body be in the horizon as at H, the angle H is called the horizontal parallax $= P'$. In this case, the angle $Z = 90°$, and the preceding formula becomes

$$\sin P' = \frac{R}{D}.$$

Substituting $\sin P'$ for $\frac{R}{D}$ in formula (a), and putting $h = SOH$, whence $Z = 90° - h$,

$$\sin P = \sin P' \cos h.$$

Since P and P' are very small, the sines are nearly proportional to the angles; hence,

$$P = P' \cos h.$$

The sun's mean horizontal parallax is about 8.75″, whence the sun's parallax in altitude may be computed.

This element is given in Table XIV.

The correction for parallax must evidently be added to the apparent altitude.

The parallax of the fixed stars is practically 0.

The Sun's Apparent Semi-Diameter.

Let C (Fig. 25) represent the centre of the earth, C' the centre of the sun, and CP a tangent to the surface of the sun.

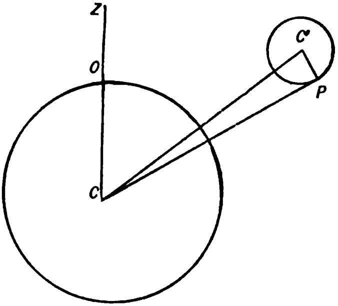

Fig. 25.

Angle PCC' is the semi-diameter of the sun as viewed from C.

Let $CC' = D$,
$OC = R$,
$PC' = r$,
angle $PCC' = S =$ the sun's semi-diameter.

In the right triangle CPC'

$$\sin S = \frac{r}{D}.$$

The formula for horizontal parallax is

$$\sin P' = \frac{R}{D}, \text{ or } D = \frac{R}{\sin P'};$$

whence, by substitution,

$$\sin S = \frac{r}{R}\sin P', \text{ or } S = \frac{r}{R}P',$$

nearly, since S and P' are very small.

Hence, the semi-diameter of the sun may be found from its horizontal parallax.

NOTE. The semi-diameter of a heavenly body, as viewed from the centre of the earth, is called the *horizontal semi-diameter.* The semi-diameter is sensibly the same whether viewed from C or O, provided the body be in the horizon, since in this position CC' and OC' are nearly equal. As the altitude of the body increases, OC' decreases, and the semi-diameter as viewed from O increases; this *augmentation of the semi-diameter* is sensible in the case of the moon, but not in the case of the sun.

The sun's semi-diameter is given for every day in the year in Table XII. It is greatest when the sun is nearest the earth, about January 1; and least when the sun is farthest from the earth, about July 1.

In order to find the true altitude of the sun's centre, the semi-diameter is added to the apparent altitude of the lower limb, and subtracted from the apparent altitude of the upper limb.

Applying the Corrections.

The *observed altitude* of the limb is given by the sextant.

The index error being applied, the result is the observed altitude above the sea horizon.

The correction for dip being applied, the result is the *apparent altitude* of the limb, referred to the true horizon.

The *true altitude* of the body is the altitude of its centre as viewed from the centre of the earth, and is obtained by applying the corrections for refraction, parallax, and semi-diameter to the apparent altitude of the limb.

The fixed stars have neither parallax nor semi-diameter.

The following symbols will be used in this chapter.

⊙ the sun.
⊖ the sun's centre.
$\overline{\odot}$ the sun's upper limb.
$\underline{\odot}$ the sun's lower limb.

EXAMPLE 1. 1882, July 5, the observed altitude of a star was 18° 20′ 30″; index correction, +1′ 17″; height of eye, 18 ft. Find the true altitude.

SOLUTION I.

Observed altitude	18° 20′ 30″
Index correction	+ 1′ 17″
	18° 21′ 47″
Dip (Table XV.)	− 4′ 9″
Apparent altitude	18° 17′ 38″
Refraction (Table XIII.)	− 2′ 54″
True altitude	18° 14′ 44″

SOLUTION II.

Observed altitude	18° 20′ 30″	Index correction	+ 1′ 17″
Correction	− 5′ 46″	Dip	− 4′ 9″
		Refraction . .	− 2′ 54″
True altitude	18° 14′ 44″		

The corrections are properly made in a certain order in the first solution; in the second solution the algebraic sum of the corrections is applied to the observed altitude. It is evident that the correction for refraction might differ by the two methods, since in one case it is based on an altitude of 18° 17′ 38″, and in the other on an altitude of 18° 20′ 30″; but in any case the difference would be so slight as to be practically unimportant in sea observations. The second method is commonly used in practice.

EXAMPLE 2. 1882, Aug. 9, the mean of observed altitudes of the sun's lower limb was 26° 0′ 2″; index correction, −0′ 17″; height of eye, 18 ft. Find the sun's true altitude.

SOLUTION I.

Observed altitude $\underline{\odot}$	26° 0′ 2″
Index correction	− 0′ 17″
	25° 59′ 45″
Dip	− 4′ 9″
Apparent altitude $\underline{\odot}$	25° 55′ 36″
Refraction − 1′ 59″ Parallax + 0′ 8″	− 1′ 51″
True altitude $\underline{\odot}$	25° 53′ 45″
Semi-diameter (Table XII.)	+ 15′ 49″
True altitude $\ominus$	26° 9′ 34″

SOLUTION II.

Observed altitude $\underline{\odot}$	26° 0′ 2″	Semi-diameter .	+ 15′ 49″
		Parallax . . .	+ 0′ 8″
Correction	+ 9′ 32″	Index correction	− 0′ 17″
		Dip	− 4′ 9″
		Refraction . .	− 1′ 59″
True altitude $\ominus$	26° 9′ 34″		

Exercise IX.

From the following data compute the true altitude:

1. 1882, April 1, obs. alt. star 25° 6′ 10″, index cor. +1′ 15″, eye 17 ft.
2. 1883, July 25, obs. alt. star 15° 20′ 25″, index cor. −2′ 20″, eye 16 ft.
3. 1882, Jan. 16, obs. alt. $\underline{\odot}$ 18° 17′ 30″, index cor. +0′ 18″, eye 18 ft.
4. 1882, Nov. 4, obs. alt. $\underline{\odot}$ 30° 12′ 40″, index cor. 0′ 0″, eye 19 ft.
5. 1882, Oct. 10, obs. alt. $\underline{\odot}$ 56° 25′ 20″, index cor. −1′ 20″, eye 17 ft.
6. 1882, Aug. 7, obs. alt. $\underline{\odot}$ 60° 10′ 10″, index cor. −2′ 15″, eye 18 ft.
7. 1882, Feb. 9, obs. alt. $\underline{\odot}$ 31° 24′ 35″, index cor. 0′ 0″, eye 18 ft.
8. 1882, Mar. 1, obs. alt. $\overline{\odot}$ 26° 17′ 20″, index cor. +2′ 15″, eye 18 ft.
9. 1882, June 18, obs. alt. $\overline{\odot}$ 20° 35′ 30″, index cor. +0′ 18″, eye 16 ft.
10. 1882, June 12, obs. alt. $\overline{\odot}$ 36° 12′ 10″, index cor. +0′ 25″, eye 20 ft.

§ 22. Time.

Apparent Solar Time is measured by the daily motion of the sun.

An **Apparent Solar Day** is the interval of time between two successive transits of the sun over the same meridian. This interval is not always the same, for two reasons:

1. The sun moves in the ecliptic which is oblique to the equator.

2. The rate of the sun's motion in the ecliptic varies, being greatest when the earth is at perihelion, and least when the earth is at aphelion.

Hence, a clock or a chronometer cannot be regulated to follow the true sun.

The **Mean Sun** is an imaginary sun supposed to move in the equator at a uniform rate equal to the mean rate of the true sun.

Mean Solar Time is the hour angle of the mean sun.

Apparent Solar Time is the hour angle of the true sun.

The **Equation of Time** is the difference between mean time and apparent time. It never exceeds about 16 minutes.

Siderial Time is measured by the apparent daily motion of the fixed stars. It is the hour angle of the vernal equinox.

The **Siderial Day** is the interval between two successive transits of the vernal equinox over the same meridian. It is about 3 min. 56 sec. shorter than the mean solar day, and is divided into 24 hours numbered from 0 to 24.

The **Civil Day** is the interval of time between two successive midnights. It is divided into two periods of 12 hours each, the first of which is marked A.M., and the second P.M.

The **Astronomical Day** begins at noon of the civil day of the same date. It is divided into 24 hours numbered from 0 to 24.

To convert civil into astronomical time:

If the given time be A.M., deduct 1 day, add 12 hours, and omit the A.M.; if the given time be P.M., simply omit the P.M.

To convert astronomical into civil time:

If the hours of the given time be less than 12, simply affix P.M.; if the given hours be 12 or more, add 1 day, subtract 12 hours, and affix A.M.

EXERCISE X.

	Astronomical Time.						*Civil Time.*						
			d.	h.	m.	s.			d.	h.	m.	s.	
1.	1882	July	8	7	6	10	Required.						
2.	1882	Mar.	7	12	25	30	Required.						
3.	1880	Jan.	1	18	10	10	Required.						
4.	1879	Dec.	31	15	0	0	Required.						
5.	1883	Feb.	2	8	4	30	Required.						
6.	Required.						1882	July	1	11	8	25	A.M.
7.	Required.						1880	Mar.	2	11	56	56	P.M.
8.	Required.						1880	Aug.	31	10	8	20	P.M.
9.	Required.						1881	Sept.	1	0	12	15	A.M.
10.	Required.						1883	Jan.	1	10	41	56	A.M.

§ 23. LONGITUDE AND TIME.

Since the earth revolves on its axis once in 24 hours, the sun appears to move over the 360° of the equator in the same time. Hence, the longitude of any place is proportional to the time required by the sun to move over an arc of the equator included between the meridian from which longitude is reckoned and the meridian of the place; and longitude may be expressed either in *time* or *arc*.

Since $360^\circ = 24$ hours,

$$\left.\begin{aligned}15^\circ &= 1\text{ hour},\\ 15' &= 1\text{ minute},\\ 15'' &= 1\text{ second},\end{aligned}\right\}\text{ or }\left\{\begin{aligned}1^\circ &= 4\text{ minutes},\\ 1' &= 4\text{ seconds},\\ 1'' &= \tfrac{1}{15}\text{ second}.\end{aligned}\right.$$

Either of the above tables enables us to convert longitude in time to longitude in arc, and *vice versa*.

§ 24. GREENWICH DATE.

The Greenwich date is the time at Greenwich corresponding to the local time elsewhere. Greenwich time should always be expressed astronomically.

To find the Greenwich date, having given the ship time and the longitude of the ship:

1. Express the ship time astronomically. (§ 22.)
2. Convert longitude in arc into longitude in time. (§ 23.)
3. If the ship be in west longitude, add the longitude in time; if the ship be in east longitude, subtract the longitude in time.

The resulting Greenwich date will be expressed in apparent time or mean time according as the local time is apparent or mean.

It is evident that local time can be found from Greenwich time by reversing these steps.

EXAMPLE 1. June 4, at 6 h. 10 m. P.M., apparent time, at ship in longitude 30° 15′ E. Find Greenwich date.

	d.	h.	m.
Ship date . . . . =	June 4	6	10
Longitude in time . . . . . . =		−2	1
Greenwich date (apparent time) =	June 4	4	9

15)30° 15′ = longitude in arc.
2 h. 1 m. = longitude in time.

EXAMPLE 2. August 21, at 8 h. 30 m. 10 s. A.M., mean time, at ship in longitude 90° 30′ W. Find Greenwich date.

		d.	h.	m.	s.
Ship date	=	Aug. 20	20	30	10
Longitude in time	=		+6	2	0
Greenwich date (mean time) .	=	Aug. 21	2	32	10

EXAMPLE 3. March 20, the sun on the meridian of the place in longitude 160° 15′ W. Find the Greenwich date.

		d.	h.	m.	s.
Ship date	=	Mar. 20	0	0	0
Longitude in time	=		+10	41	0
Greenwich date (apparent time)	=	Mar. 20	10	41	0

EXERCISE XI.

	Local Civil Date.						*Longitude.*	*Greenwich Date.*				
		d.	h.	m.	s.							
1.	May	4	6	12	15	P.M.	170° 50′ 0″ W.	Required.				
2.	July	31	11	12	30	A.M.	40° 20′ 0″ W.	Required.				
3.	Aug.	1	2	10	15	A.M.	80° 40′ 45″ W.	Required.				
4.	Mar.	2	10	20	0	P.M.	50° 45′ 0″ E.	Required.				
5.	Mar.	25	11	8	0	P.M.	100° 25′ 30″ W.	Required.				
									d.	h.	m.	s.
6.	Required.						25° 7′ 0″ W.	Dec.	30	19	47	28
7.	Required.						179° 0′ 0″ W.	July	4	23	51	0
8.	Required.						179° 0′ 0″ E.	July	3	23	59	0
9.	Required.						45′ 0″ E.	May	19	19	40	20
10.	Required.						2° 10′ 0″ E.	1880. Dec.	31	15	8	0

§ 25. THE CHRONOMETER.

A chronometer is a time-piece adapted to use on board ship. Unless otherwise specified, chronometers are supposed to be set to Greenwich mean time.

The error of chronometer on mean time at any place is the difference between the mean time at that place and the time indicated by the chronometer.

The Error of Chronometer on mean time at Greenwich is the difference between mean time at Greenwich and the time by chronometer. The error is *fast* or *slow* according as the chronometer is in advance of or behind the mean time at Greenwich.

The **Rate of Chronometer** is the daily change in its error; it is *gaining* or *losing* according as the chronometer is going too fast or too slow.

To determine the error of chronometer on Greenwich mean time the longitude of the place must be known, whence the Greenwich mean time may be found (§ 24); this compared with the chronometer set to Greenwich time gives the error.

The rate may be found by determining the error on different days, and noting the daily change, supposing it to be uniform.

EXAMPLE 1. February 12, the error of chronometer was 9 m. 19.6 s. slow, and February 25, the error was 10 m. 30.2 s. slow. Find the rate.

	m.	s.
Feb. 12, slow	9	19.6
Feb. 25, slow	10	30.2
Change for 13 days	1	10.6
	60	
	13)70.6	
Change for 1 day, or rate	5.4 s. losing.	

Since the chronometer was slow, and the error increasing, the rate was losing.

EXAMPLE 2. 1882, April 8, P.M., an observation was made, when the time was April 8, 4 h. 10 m. 22 s. by chronometer, which was 18 m. 10 s. fast on Greenwich mean time on Jan. 1, 1882, and 16 m. 17 s. fast on January 31. Find the Greenwich date by chronometer.

	m.	s.
Jan. 1, fast	18	10
Jan. 31, fast	16	17
Change for 30 days	1	53
	60	
	30)113	
Rate	3.8 s. losing.	

Since the chronometer was fast, and the error decreasing, the rate was losing.

	d. h. m. s.	d. h.	s.	
Time by chronometer, April	8 4 10 22	Feb. 28	3.8	
Original error . .	−16 17	Mar. 31	228.0	60
	8 3 53 5	April 8 4	26.6	7
Accumulated rate .	+4 15.3	67 4	.38	.1
Greenwich date . . April	8 3 57 20.3	67.17	.27	.07
			255.25	

255.25 s. = 4 m. 15.3 s.

The error on Jan. 31 is 16 m. 17 s. fast, which we deduct. From Jan. 31 to April 8 d. 3 h. 57 m. is nearly 67 d. 4 h. = 67.17 d. The error for one day is 3.8 s.; hence, for 67.17 d. the error is 3.8 s. × 67.17 = 255.3 s. = 4 m. 15.3 s. This accumulated rate (4 m. 15.3 s.) must be added, and the result is the required Greenwich date.

The method of multiplying 3.8 s. by 67.17, employed above (at the right of the page), will be made plain by the arrangement of the work shown in the margin.

3.8 s.

$228.0 = 3.8 \times 60$

$26.6 = 3.8 \times 7$

$.38 = 3.8 \times .1$

$.27 = 3.8 \times .07$

$255.25 = 3.8 \times 67.17$

§ 26. The Nautical Almanac.

The American and British Nautical Almanacs give, at equidistant instants of Greenwich time, the apparent right ascensions and declinations of the sun, moon, planets, and certain fixed stars; the semi-diameters of the sun and moon; the equation of time; etc., etc.

Table XII. contains the data from the Nautical Almanac which are necessary for the solution of the exercises in this chapter.

To reduce the sun's declination by means of the hourly difference:

Table XII. gives the sun's declination at apparent and mean noon, Greenwich time, and the hourly change in declination for every day in the year.

If the declination at noon be required, it may be taken at once from the table. The declination at any other time may be found by simple interpolation; that is, by assuming that the declination varies as the time. The column headed "Diff. for 1 hour," contains the average change for one hour. This multiplied by the number of hours gives the correction required.

NOTE. In Table XII. the difference of declination for 1 hour is marked + or − according as the sun is moving northward or southward, but in the following examples "Diff. for 1 h." is marked + or − according as the element to which it is to be applied is increasing or decreasing numerically.

EXAMPLE 1. Find the sun's declination for 1882, Jan. 7 d. 10 h., apparent time, Greenwich date.

⊙'s dec. Jan. 7 d. 0 h. = 22° 21′ 5.6″ S.	Diff. for 1 h. = − 19.41″
− 3′ 14.1″	10
⊙'s dec. Jan. 7 d. 10 h. = 22° 17′ 51.5″ S.	Diff. for 10 h. = − 194.1″
	= − 3′ 14.1″

The table gives the declination for Jan. 7 d. 0 h. = 22° 21′ 5.6″ S., and the hourly difference = 19.41″, which is marked − because the declination is decreasing. For 10 h. the declination is 19.41″ × 10 = 194.1″ = 3′ 14.1″, which must be subtracted from the declination at noon.

EXAMPLE 2. Find the sun's declination for 1882, Feb. 12 d. 17 h. 12 m. 10 s., mean time, Greenwich date.

⊙'s dec. Feb. 13 d. 0 h. = 13° 17′ 9.1″ S.	Diff. for 1 h. = −50.59″
5′ 43.9″	6.8
⊙'s dec. Feb. 12 d. 17 h.	Diff. for 6.8 h. = 343.912″
12 m. 10 s. = 13° 22′ 52.0″ S.	= 5′ 43.9″

In this example, the given date is nearer Feb. 13 d. 0 h. than Feb. 12 d. 0 h.; hence, we take from the table the declination for the former date. Feb. 13 d. 0 h. − Feb. 12 d. 17 h. 12 m. 10 s. = 6 h. 47 m. 50 s. = 6.8 h., for which the difference is found to be 5′ 43.9″, which evidently must be added to the declination at noon.

EXAMPLE 3. 1882, May 5, apparent noon, longitude 57° W.; required the sun's declination.

		d.	h.	m.
Local time	= May	5	0	0
Longitude in time =		0	3	48
Greenwich date	= May	5	3	48

☉'s dec. May 5 d. 0 h.	= 16° 18′ 5.9″	Diff. for 1 h. =	+ 42.66″
	+ 2′ 42.1″		3.8
☉'s dec. May 5 d. 3 h. 48 m.	= 16° 20′ 48″	Diff. for 3.8 h. =	162.108″
			= + 2′ 42.1″

To reduce the equation of time.

EXAMPLE 4. Greenwich date, mean time, 1882, March 2 d. 3 h. 10 m.; find the equation of time.

	m. s.		s.
Eq. of time Mar. 2 d. 0 h.	= − 12 18.65	Diff. for 1 h. =	− 0.524
	− 1.66		$3.1\frac{2}{3}$
Eq. of time Mar. 2 d. 3 h. 10 m. =	12 16.99	Diff. for $3.1\frac{2}{3}$ h. =	− 1.6593

Table XII. gives the equation of time corresponding to mean time, Mar. 2 d. 0 h. = 12 m. 18.65 s. The − sign indicates that the equation of time is to be subtracted from mean time to obtain apparent time.

The difference for one hour is also found in the table and marked −, because the equation of time is decreasing. The difference for 3 h. 10 m. is found to be 1.66 s., which is subtracted from the equation of time at noon.

This method is not strictly correct, because the change in declination is not uniform; but it will be found accurate enough in most cases, and will be used in this chapter.

EXERCISE XII.

Find the sun's declination and the equation of time corresponding to the following Greenwich dates:

			d.	h.	m.	s.	
1.	1882	Jan.	7	3	0	0	apparent time.
2.	1882	Aug.	1	6	12	20	apparent time.
3.	1882	May	5	10	25	0	apparent time.
4.	1882	Aug.	7	15	12	0	apparent time.
5.	1882	Dec.	4	6	18	0	apparent time.
6.	1882	July	23	20	16	40	apparent time.
7.	1882	Nov.	1	3	6	0	apparent time.
8.	1882	Oct.	12	5	12	0	apparent time.
9.	1882	June	7	3	18	0	mean time.
10.	1882	Feb.	3	9	15	0	mean time.

§ 27. To find the Hour Angle of a Heavenly Body.

Let $t =$ the hour angle of the heavenly body.
$L =$ the latitude of the observer.
$d =$ the declination of the body.
$h =$ the true altitude of the body.
$p = 90^\circ \pm d =$ the polar distance of the body.
$S = \frac{1}{2}(L + p + h)$.
$R = \frac{1}{2}(L + p - h) = \frac{1}{2}(L + p + h) - h = S - h$.

The general formula for the hour angle of a heavenly body in terms of its polar distance and altitude, and the latitude of the observer, is

$$\sin \tfrac{1}{2} t = \pm \left[\cos S \sin R \sec L \csc p\right]^{\frac{1}{2}}. \qquad \text{[Spher. Trig. § 67.]}$$

Example 1. The true altitude of a heavenly body is 14° 50′ 42″; its declination, 16° 20′ 5″ S.; and the latitude of the observer, 37° N. Find the hour angle.

$$\begin{array}{rlr}
h = & 14^\circ 50\ 42'' & \\
L = & 37^\circ & \log\sec 0.09765 \\
p = & 106^\circ 20'\ 5'' & \log\csc 0.01789 \\
\cline{2-2}
2S = & 158^\circ 10' 47'' & \\
S = & 79^\circ\ 5' 24'' & \log\cos 9.27708 \\
R = & 64^\circ 14' 42'' & \log\sin 9.95456 \\
\cline{3-3}
 & & 2)\overline{19.34718} \\
 & & \log\sin \tfrac{1}{2}t = 9.67359 \\
 & & \therefore \tfrac{1}{2}t = 28^\circ\ 8' 23'' \\
 & & t = 56^\circ 16\ 46'' \\
 & & = 3\text{ h. } 45\text{ m. } 7\text{ s.}
\end{array}$$

§ 28. To find the Local Time.

The local apparent time is the hour angle of the true sun, if the sun is west of the meridian; if the sun is east of the meridian, the local apparent time is the difference between 24 hours and the hour angle.

EXAMPLE 1. On board the U.S. Steamer *Swatara*, in latitude 18° 12′ S., Greenwich mean time, 1882, Sept. 11 d. 14 h. 55 m. 50 s.; observed altitude of $\underline{\odot}$, 27° 18′ 25″; index error + 31″; height of eye, 18 ft. Required the local mean time.

Sept. 11 d. 14 h. 55 m. 50 s. = 9 h. 4 m. 10 s. (= 9.07 h.) before noon, Sept. 12.

To reduce the sun's declination and find the polar distance:

$\odot$'s dec. Sept. 12 d. 0 h. =	4° 7′ 20.8″ N.	Diff. for 1 h. =	−57.36″
Change for 9.07 h. =	8′ 40.3″	Diff. for 9 h. =	516.24″
$\odot$'s dec. Sept. 11 d. 14 h.		Diff. for .07 h. =	4.02″
55 m. 50 s. =	4° 16′ 1″ N.		520.26″
	90°	Diff. for 9.07 h. =	8′ 40.3″
p =	94° 16′ 1″		

To find the true altitude:

Obs. alt. $\underline{\odot}$,	26° 18′ 20″	semi-diam.,	+ 15′ 56″
		index error,	+ 31″
	+ 10′ 29″	parallax,	+ 8″
		dip,	− 4′ 9″
		refraction,	− 1′ 57″
True altitude,	26° 28′ 49″		

To reduce the equation of time:

	m. s.		s.
Eq. of time Sept. 12 d. 0 h.	= 3 48.62	Diff. for 1 h. =	+0.871
Change for 9.07 h.	7.90	Diff. for 9 h. =	7.839
Eq. of time Sept. 11 d. 14 h.		Diff. for .07 h. =	.061
55 m. 50 s.	= 3 40.7	Diff. for 9.07 h. =	7.900

To find the hour angle of the sun:

$$\sin \tfrac{1}{2} t = \pm \left[\cos S \sin R \sec L \csc p\right]^{\frac{1}{2}}.$$

h =	26° 28′ 49″		
L =	18° 12′ 0″	log sec =	0.02229
p =	94° 16′ 1″	log csc =	0.00121
$2S$ =	138° 56′ 50″		
S =	69° 28′ 25″	log cos =	9.54486
R =	42° 59′ 36″	log sin =	9.83373
		2	19.40209
		log sin $\frac{1}{2}t$ =	9.70105
		∴ $\frac{1}{2}t$ =	30° 9′ 33″
		t =	60° 19′ 6″

		h.	m.	s.
$t = 60° 19' 6''$	=	4	1	16
		24		
Local apparent astronomical time	= Sept. 11,	19	58	44
Equation of time	=		−3	41
Local mean astronomical time	= Sept. 11,	19	55	3
Local mean civil time	= Sept. 12,	7	55	3 A.M.

If the time be A.M., apparent civil time = 12 h. − t.

EXAMPLE 2. 1882, Aug. 23, P.M., at sea, in latitude 9° 37′ N. Five observed altitudes of the ☉ were taken at the times (by the watch), standing opposite, viz.:

		h.	m.	s.
Obs. alt. ☉	21° 33′ 30″	2	42	58
	21° 28′ 10″	2	43	21.5
	21° 24′ 00″	2	43	39
	21° 19′ 10″	2	43	59.5
	21° 14′ 30″	2	44	18.5

Index error, 0; height of eye, 18 ft.; correction of watch by chronometer, −5 h. 16 m. 49.3 s. Required the local time.

The mean of the five observed altitudes is found by dividing their sum by 5, and the mean time (by a watch) of these observations is found in a similar manner. The Greenwich mean time is found to be Aug. 22 d. 21 h. 26 m. 50 s., which is 2 h. 33 m. 10 s. before noon, Aug. 23.

Obs. alt. ☉.	h.	m.	s.
21° 33′ 30″	2	42	58
21° 28′ 10″	2	43	21.5
21° 24′ 00″	2	43	39
21° 19′ 10″	2	43	59.5
21° 14′ 30″	2	44	18.5
119′ 20″		218	16.5
21° 23′ 52″	2	43	39.3
	−5	16	49.3
	9	26	50
	21	26	50 G.M.T.
	2	33	10 = 2.55 h.

☉'s dec.
11° 24′ 14.4″ N.
+ 2′ 10.0″
11° 26′ 24.4″ N.
90°
78° 33′ 36″

Diff. for 1 h.	= + 50.97″
Diff. for 2 h.	= 101.94″
Diff. for .5 h.	= 25.48″
Diff. for .05 h.	= 2.55″
	129.97″
Diff. for 2.55 h.	= 2′ 10″
Semi-diam.,	+ 15′ 52″
Par.,	+ 8″
Dip,	− 4′ 9″
Ref.,	− 2′ 28″
	+ 9′ 23″

$$\begin{array}{rlrl}
 & \underline{9' \, 23''} & & \\
h = & 21^\circ \, 33' \, 15'' & & \\
L = & 9^\circ \, 37' \, 00'' & \log \sec = & 0.00615 \\
p = & \underline{78^\circ \, 33' \, 36''} & \log \csc = & 0.00871 \\
2S = & 109^\circ \, 43' \, 51'' & & \\
S = & 54^\circ \, 51' \, 56'' & \log \cos = & 9.76004 \\
R = & 33^\circ \, 18' \, 41' & \log \sin = & 9.73972 \\
 & & & 2\overline{)19.51462} \\
 & & \log \sin \frac{1}{2} t = & 9.75731
\end{array}$$

$$\begin{array}{rl}
 & \text{h. m. s.} \\
\therefore t = & 4 \;\; 39 \;\; 3 \\
\text{Equation of time} = & \underline{+ 2 \;\; 30} \\
\text{Local mean time} = & 4 \;\; 41 \;\; 33 \text{ P.M.}
\end{array}$$

Eq. of time.	Diff. for 1 h.	
m. s.	s.	
2 28.19	– 0.648	
1.65	1.296	2
2 29.84	.324	.5
	.032	.05
	1.652	

§ 29. LATITUDE BY MERIDIAN OBSERVATION OF A HEAVENLY BODY.

Let HZH' (Fig. 25) represent the meridian of the observer, P the nearest pole, Z the zenith, HH' the horizon, EQ the equator, and S the heavenly body on the meridian.

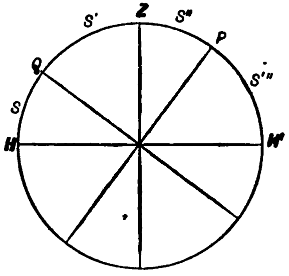

Fig. 25.

$QS = d =$ the declination of the body.

$HS = h =$ the true altitude of the body.

$SZ = z = 90^\circ - h =$ the zenith distance of the body (named N. or S. according as the zenith is north or south of the body).

$QZ = PH' = L =$ the latitude of the observer.

$QZ = SZ + QS$, or $L = z + d$.

If the body be at S',

$$QZ = S'Z - QS', \text{ or } L = z - d.$$

If the body be at S'',

$$QZ = QS'' - ZS'', \text{ or } L = d - z$$

If the body be at S''',

$$QZ = QS''' - ZS''', \text{ or } L = d - z.$$

Hence, to find the latitude of a place by meridian observation of a heavenly body,

If the declination and zenith distance of the body are of the same name, take their arithmetical sum; if of different names, take their arithmetical difference and give it the name (N. or S.) of the greater.

(a) *Latitude by meridian altitude of the sun.*

Example. 1882, Aug. 11, in longitude 92° 12′ E., the observed meridian altitude of the sun's lower limb was 42° 42′ 10″ N.; index correction, − 2′ 50″; height of the eye, 17 ft.; required the latitude.

Since the sun bears *north*, the zenith distance is *south*.

		d.	h.	m.	s.
Ship date,	Aug.	11	0	0	0
Longitude in time,			6	8	48
Greenwich date,	Aug.	10	17	51	12

☉'s dec. Aug. 11 d. 0 h.	= 15° 14′ 24″ N.	− 44.61
	4′ 34″	6.15
☉'s dec. Aug. 10 d. 17 h. 51 m. 12 s.	= 15° 18′ 58″ N.	274.35

Observed altitude $\underline{\odot}$,	42° 42′ 10″
Index correction,	− 2′ 50″
	42° 39′ 20″
Dip,	− 4′ 2″
Apparent altitude $\underline{\odot}$,	42° 35′ 18″
Refraction,	− 1′ 3″
	42° 34′ 15″
Parallax,	+ 7″
True altitude $\underline{\odot}$,	42° 34′ 22″

True altitude $\underline{\odot}$,	42° 34′ 22″
Semi-diameter,	+ 15′ 50″
True altitude,	42° 50′ 12″
	90°
Zenith distance,	47° 9′ 48″ S.
Declination,	15° 18′ 58″ N.
Latitude,	31° 50′ 50″ S.

The work may be arranged more compactly as follows:

Long. 92° 12′ E. = 6 h. 8 m. 48 s.

$\underline{\odot}$ 42° 42′ 10″	Index cor., − 2′ 50″		
	Semi-diam., + 15′ 50″	☉'s dec. 15° 14′ 24″ N.	44.61
+ 8′ 2″	Dip, − 4′ 2″	4′ 34″	6.15
	Ref., − 1′ 3″	d = 15° 18′ 58″ N.	274.35
	Par., + 7″	z = 47° 9′ 48″ S.	
42° 50′ 12″		L = 31° 50′ 50″ S.	
90°			
z = 47° 9′ 48″ S.			

EXERCISE XIII.

From the following data find the latitude:

	Civil Date.	Longitude.	Obs. Merid. Alt. $\underline{\odot}$	Index Cor.	Eye.
1.	1882, Jan. 1.	102° 41′ W.	59° 59′ 50″ S.	+ 0′ 50″	15
2.	1882, Feb. 1.	78° 14′ E.	78° 4′ 10″ S.	+ 0′ 55″	12
3.	1882, Mar. 20.	173° 18′ W.	89° 37′ 0″ N.	+ 4′ 32″	18
4.	1882, April 1.	87° 42′ W.	48° 42′ 30″ S.	+ 1′ 42″	18
5.	1882, Sept. 1.	97° 42′ E.	51° 4′ 50″ S.	− 6′ 0″	23
6.	1882, Aug. 26.	92° 3′ E.	35° 35′ 20″ N.	+ 2′ 17″	12
7.	1882, May 16.	45° 26′ W.	86° 34′ 20″ N.	+ 4′ 16″	15
8.	1882, Mar. 20.	174° 0′ W.	89° 56′ 10″ N.	− 1′ 15″	15
9.	1882, June 1.	44° 40′ E.	72° 14′ 10″ N.	+ 3′ 45″	22
10.	1882, Dec. 1.	67° 56′ E.	18° 48′ 10″ S.	− 3′ 6″	18
11.	1882, Sept. 23.	57° 45′ E.	84° 10′ 50″ N.	− 1′ 36″	16
12.	1882, Sept. 23.	119° 54′ E.	83° 46′ 0″ S.	− 5′ 30″	18
13.	1882, Nov. 21.	70° 20′ E.	80° 20′ 0″ N.	− 2′ 50″	20
14.	1882, Dec. 31.	123° 45′ W.	67° 8′ 10″ S.	+ 0′ 9″	13
15.	1882, Oct. 20.	150° 25′ W.	49° 58′ 50″ N.	+ 1′ 10″	19
16.	1882, June 1.	96° 17′ E.	75° 38′ 15″ N.	+ 0′ 27″	26
17.	1882, June 25.	59° 15′ E.	Obs. alt. $\overline{\odot}$. 60° 23′ 15″ N.	+ 2′ 21″	30

(b) *Latitude by meridian altitude of a fixed star.*

Since the declination of a fixed star changes very slowly, it is sensibly the same for several successive days; hence, the Greenwich date and the longitude are not required.

Example. 1882, Dec. 21, the observed meridian altitude of the star Procyon (declination 5° 31′ 16″ N.) was 52° 51′ 50″ N.; index correction, −49″; height of eye, 21 ft.; required the latitude.

$$\begin{array}{ll}
\text{Obs. alt.,} & 52^\circ\ 51'\ 50''\ \text{N.} \\
 & -6'\ \ 2'' \left\{\begin{array}{ll}\text{Index cor.,} & -0'\ 49'' \\ \text{Dip,} & -4'\ 29'' \\ \text{Ref.,} & -0'\ 44''\end{array}\right. \\
\hline
\text{True alt.,} & 52^\circ\ 45'\ 48'' \\
 & 90 \\
\hline
\text{Zenith dist.,} & 37^\circ\ 14'\ 12''\ \text{S.} \\
\text{Declination,} & 5^\circ\ 31'\ 16''\ \text{N.} \\
\hline
\text{Latitude,} & 31^\circ\ 42'\ 56''\ \text{S.}
\end{array}$$

The following table gives the declinations necessary for Exercise XIV.

Declinations of Fixed Stars.

Date.	*Star.*	*Declination.*
1882, Jan. 24.	Aldebaran.	16° 16′ 14″ N.
1882, Feb. 1.	Sirius.	16° 33′ 32″ S.
1882, Feb. 12.	Procyon.	5° 31′ 20″ N.
1882, Mar. 19.	Arcturus.	19° 47′ 33″ N.
1882, Mar. 31.	Spica.	10° 32′ 59″ S.
1882, April 11.	Sirius.	16° 33′ 38″ S.
1882, July 6.	Antares.	26° 10′ 13″ S.
1882, Aug. 20.	Altair.	8° 33′ 49″ N.
1882, Aug. 20.	β Centauri.	59° 48′ 36″ S.
1882, Sept. 7.	Arcturus.	19° 47′ 49″ N.
1882, Oct. 7.	Markab.	14° 34′ 46″ N.
1882, Nov. 7.	Fomalhaut.	30° 14′ 33″ S.
1882, Dec. 7.	α Arietis.	22° 54′ 38″ N.
1882, Dec. 21.	Procyon.	5° 31′ 16″ N.

Exercise XIV.

From the following data find the latitude:

	Civil Date.	*Star.*	*Obs. Merid. Alt.*	*Index Cor.*	*Eye.*
1.	1882, Jan. 24.	Aldebaran.	52° 36′ 0″ S.	− 0′ 23″	20
2.	1882, Feb. 12.	Procyon.	77° 18′ 10″ S.	+ 0′ 19″	16
3.	1882, Mar. 19.	Arcturus.	36° 10′ 20″ N.	+ 2′ 42″	20
4.	1882, Aug. 20.	Altair.	66° 51′ 10″ N.	+ 0′ 58″	13
5.	1882, Nov. 7.	Fomalhaut.	59° 40′ 0″ N.	+ 1′ 12″	23
6.	1882, Sept. 7.	Arcturus.	86° 35′ 50″ N.	− 1′ 10″	12
7.	1882, Oct. 7.	Markab.	54° 10′ 15″ S.	0	13
8.	1882, Aug. 20.	β Centauri.	59° 47′ 13″ S.	0	25
9.	1882, Dec. 7.	α Arietis.	60° 29′ 50″ S.	− 2′ 10″	18
10.	1882, Feb. 1.	Sirius.	37° 50′ 20″ S.	+ 1′ 4″	19
11.	1882, April 11.	Sirius.	61° 3′ 50″ N.	0	16
12.	1882, Mar. 31.	Spica.	52° 14′ 0″ N.	0	19
13.	1882, July 6.	Antares.	70° 10′ 30″ N.	0	21

§ 30. Latitude by Ex-Meridian Altitude of the Sun, by Reduction to the Meridian.

If it happens that the meridian altitude of the sun cannot be observed, owing to clouds or other causes, the altitude may be observed before or after noon and reduced to the meridian altitude.

Towson's Method.

Towson's Tables for the reduction of ex-meridian altitudes furnish a simple practical solution of the problem under consideration.

Corresponding to the nearest hour angle (§ 27) and nearest declination (§ 26) of the sun at the time of observation, Towson's first table gives an *Index Number* and *Augmentation I.* The *declination of the sun at noon* is found by adding Augmentation I. to the declination at the time of observation.

Corresponding to the Index Number (given by the first table) and the nearest true altitude of the sun at the time of observation, Towson's second table gives Augmentation II.

The *true altitude of the sun at noon* is found by adding Augmentation II. to the true altitude at the time of observation.

Having the declination and altitude at noon, the latitude is found as in § 29.

The latitude may be found from these tables by ex-meridian altitudes of a star or planet, but siderial time must be employed in finding the hour angle.

If equal altitudes be observed before and after meridian passage, the hour angle may be found by taking half of the elapsed time; or, if the two altitudes differ only a few minutes (of arc) their mean may be reduced by employing half the elapsed time as the hour angle.

It should be observed that Towson's method is independent of the latitude by account.

The maximum altitude in Towson's Tables is 74°, and the hour angle corresponding cannot exceed 22 minutes. The greatest declination is 23° 20′.

Towson's method will be illustrated by a single example.

EXAMPLE. 1882, Dec. 2, P.M., at ship, in longitude 4° 39′ W.; observed altitude ☉ south of observer was 24° 14′ 10″; height of eye, 11 ft.; time by watch, Dec. 2d. 0h. 50m. 58s. which had been found to be 19m. 38s. fast on apparent time at ship; the difference of longitude made to the west was 21.3′, after the error on apparent time was determined. Required the latitude.

	d.	h.	m.	s.
Time by watch, Dec.	2	0	50	58
Watch fast,			−19	38
	2	0	31	20
Diff. long. $\frac{21.3}{15}$,			−1	25
App. time at ship, Dec.	2	0	29	55
Time from noon,			29	55

	d.	h.	m.	s.
App. time at ship, Dec.	2	0	29	55
Longitude, 4° 39′ W.			+18	36
Greenwich date, Dec.	2	0	48	31

48m 31s. = .81h.

Dec. 2d. 0h.,	21° 59′ 53.1″	23.13″
	18.7″	.81″
Reduced declination,	22° 0′ 12″	18.74″
Augment. (Towson, Table I.),	+10′ 17″	
Meridian declination,	22° 10′ 29″	

Observed altitude ☉,	24° 14′ 10″	
	+11′ 0″	Semi-diam., +16′ 16″ Dip, − 3′ 15″ Parallax, + 0′ 8″ Refraction, − 2′ 9″
True altitude,	24° 25′ 10″	
Aug. (Towson, Table III., index 75),	+11′ 56″	
Meridian altitude,	24° 37′ 6″	
	90°	
Zenith distance,	65° 22′ 54″	
Meridian declination,	22° 10′ 29″	
Latitude,	43° 12′ 25″	

§ 31. LONGITUDE BY CHRONOMETER, FROM AN OBSERVED ALTITUDE OF THE SUN.*

The difference of longitude of two places is equal to the difference of time reduced to arc (§ 23).

Hence, the longitude of a place from Greenwich may be determined by taking the difference between the local mean time and the Greenwich mean time, and reducing to arc. The longitude will be east or west according as the Greenwich time is less or greater than local time.

The local mean time is found as in § 28. The Greenwich mean time is found by chronometer set to Greenwich time. (See § 25; also, the example of § 30.)

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* As an astronomical question, the determination of longitude resolves itself into the determination of the difference of time *reckoned at the two meridians at the same* ABSOLUTE *instant.* For seamen, the only *really* practical methods of effecting this are: first by the chronometer, and secondly by lunars. These last, however, are rapidly dying out, and are mostly looked upon now as "fancy navigation." . . . In the class of vessels most likely to need lunars (namely, those small craft which, for the sake of economy, carry but one chronometer), it is not likely than an expensive sextant or quintant will be found; and if by chance it were, it is questionable whether the requisite expertness in observing and calculating would accompany it. Therefore, *practically speaking,* lunars are out of date. — *Captain Lecky in "'Wrinkles' in Practical Navigation,"* p. 175.

EXAMPLE. 1. 1882, February 10, A.M., at ship, latitude 50° 48′ N.; observed altitude $\underline{\odot}$ 9° 10′ 50″; index correction, − 3′ 18″; height of eye, 18 ft.; time by chronometer, February 9 d. 9 h. 59 m. 25 s., which was 37 m. 58.8 s. fast for mean noon at Greenwich, December 20, 1881, and on January 10, 1882, was 34 m. 12 s. fast for mean noon at Greenwich; from the time of observation until noon the ship has sailed on an S.W. course (true) 34 miles. Required the longitude at noon.

	m.	s.
December 20, chronometer fast,	37	58.8
January 10, chronometer fast,	34	12.0
Change in 21 days,	3	46.8
	60	
	226.8	

21 { 3)226.8
7)75.6
10.8 losing.
324.0 30.
4.3 .4
60)328.3
5 28.3

The change in 1 d. is 226.8 s. ÷ 21 = 10.8 s.; the change in 30.4 d. (the interval between noon January 10, and February 9 d. 10 h.) is 10.8 s. × 30.4 = 328.3 s. = 5 m. 28.3 s.

		d.	h.	m.	s.
Time by chronometer,	Feb.	9	9	59	25
Original error,				− 34	12
		9	9	25	13
Accumulated rate,				+ 5	28
Greenwich mean time,		9	9	30	41

Time, after noon 9 h. 31 m. = 9.52 h.

⊙'s declination, 14° 36′ 17.7″ S.
7′ 39.6″
14° 28′ 38.1″
90°
p = 104° 28′ 38.1″

− 48.28″
9.52″
459.63″
7′ 39.6″.

Obs. alt. $\underline{\odot}$,	9° 10′ 50″	Semi-diam.,	+ 16′ 15″
Correction,	+ 3′ 10″	Parallax,	+ 9″
True alt. ⊙,	9° 14′ 0″		

Ind. cor., − 3′ 18″
Dip, − 4′ 9″
Refract., − 5′ 47″

$h =$ 9° 14′ 0″		
$L =$ 50° 48′ 0″	log sec	0.19926
$p =$ 104° 28′ 38″	log csc	0.01402
$2S =$ 164° 30′ 38″		
$S =$ 82° 15′ 19″	log cos	9.12956
$R =$ 73° 1′ 19″	log sin	9.98064
		2)19.32348

Eq. of time.

m. s.	s.
14 26.65	+0.052
+ .50	9.520
14 27.15	.495

$\log \sin \tfrac{1}{2} t = 9.66174$

$\therefore \tfrac{1}{2} t = 27° 19′ 3″$

$t = 54° 38′ 6″$

= 3h. 38m. 32s. Time before noon, Feb. 10.

	d.	h.	m.	s.	
Feb.	9	20	21	28	Local apparent astronomical time.
		+	14	27	Equation of time.
Feb.	9	20	35	55	Local mean astronomical time.
Feb.	9	9	30	41	Greenwich mean time.
		11	5	14	Difference of time.

166° 18′ 30″ E. long. at time of observation.

Longitude at sights,	166° 19′ E.	Latitude at sights,	50° 48′ N.
Change until noon,	38′ W.	Change until noon,	24′ S.
Longitude at noon,	165° 41′ E.	Latitude at noon,	50° 24′ N.

From the time of observation until noon the ship sailed S.W. 34 miles. The difference of latitude and of longitude corresponding to this course and distance are found by Middle Latitude Sailing to be 24′ S. and 38′ W. respectively.

EXAMPLE 2. 1882, November 8, P.M., at sea, on board the U.S. Steamer *Swatara*, in latitude 30′ S.; five observed altitudes of the ☉ were taken at the times (by watch) standing opposite, viz.:

		h.	m.	s.
Obs. alt. ☉,	24° 51′ 50″	11	8	3
	48′ 50″			16
	44′ 40″			33
	40′ 20″			49
	37′ 0″			64.5

Index error, +30″; height of eye, 18 ft.; correction of watch by chronometer, −5h. 12m. 15.9s. Required the longitude.

	h. m. s.		
24° 51′ 50″	11 8 3	⊙'s declination.	
48′ 50″	16	16° 38′ 20.6″ S.	+ 43.50″
44′ 40″	33	+ 4′ 18.4″	5.94″
40′ 20″	49	16° 42′ 39.0″	258.37″
37′ 0″	64.5	90°	= 4′ 18.4″
222′ 40″	165.5	73° 17′ 21″	
24° 44′ 32″	11 8 33.1	Semi-diameter,	+ 16′ 12″
	− 5 12 15.9	Parallax,	+ 8″
	5 56 17.2	Index correction,	+ 30″
	= 5.94 h. P.M.	Dip,	− 4′ 9″
		Refraction,	− 2′ 6″
10′ 35″			+ 10′ 35″

h = 24° 55′ 7″		Eq. of time.	
L = 0° 30′ 0″	log sec 0.00002	m. s.	s.
p = 73° 17′ 21″	log csc 0.01874	16 7.21	−.191
$2S$ = 98° 42′ 28″		− 1.13	5.94
S = 49° 21′ 14″	log cos 9.81383	16 6.08	1.13
R = 24° 26′ 7″	log sin 9.61665		
	2)19.44924		

log sin ½ t = 9.72462

∴ t = 4 h. 16 m. 16 s. Time after noon, Nov. 8.

d. h. m. s.	
Nov. 8 4 16 16	Local apparent astronomical time.
16 6	Equation of time.
Nov. 8 4 0 10	Local mean astronomical time.
Nov. 8 5 56 17	Greenwich mean time.
1 56 7	Difference of time.
29° 1′ 45″	W. Longitude.

Exercise XV.

1. 1882, October 19, A.M., at sea, on board the U.S. Steamer *Swatara*, in latitude 33° 27′ S.; the observed altitude ⊙̲, 28° 22′ 30″; index correction, +30″; height of eye, 18 ft.; Greenwich mean time by chronometer, October 18 d. 18 h. 28 m. 38 s. Required the longitude.

2. 1882, October 20, A.M., at sea, on board the *Swatara*, in latitude 31° 40′ S.; the observed altitude $\underline{\odot}$, 35° 16′ 10″; index correction, +30″; height of eye, 18 ft.; Greenwich mean time by chronometer, October 19 d. 19 h. 11 m. 24 s. Required the longitude.

3. 1882, October 20, P.M., at sea, on board the *Swatara*, in latitude 30° 55′ S.; the observed altitude $\underline{\odot}$, 21° 42′ 30″; index correction, +29″; height of eye, 18 ft.; Greenwich mean time by chronometer, October 20 d. 3 h. 35 m. 40 s. Required the longitude.

4. 1882, October 21, A.M., at sea, on board the *Swatara*, in latitude 29° 35′ S.; the observed altitude $\underline{\odot}$, 24° 26′ 42″; index correction, +29″; height of eye, 18 ft.; Greenwich mean time by chronometer, October 20 d. 18 h. 30 m. 39 s. Required the longitude.

5. 1882, January 29, P.M., at ship, latitude 42° 26′ N.; observed altitude $\underline{\odot}$, 13° 40′; index error, −1′ 8″; height of eye, 16 ft.; time by chronometer, 29 d. 6 h. 48 m. 40 s., which was slow 11 m. 22.3 s. for mean noon at Greenwich, December 1, 1881, and on January 1, 1882, was 8 m. 7 s. slow for Greenwich mean noon. Required the longitude.

6. 1882, March 31, A.M., at ship, latitude 26° 9′ N.; observed altitude $\underline{\odot}$, 29° 10′ 20″; height of eye, 26 ft.; time by chronometer, 31 d. 0 h. 4 m. 50 s., which was 58 m. 58 s. fast for mean noon at Greenwich, November 20, 1881, and on December 31, 1881, was 1 h. 2 m. 55.8 s. fast for mean time at Greenwich. Required the longitude.

7. 1882, May 22, A.M., at ship, latitude 43° 25′ N.; observed altitude $\underline{\odot}$, 32° 8′; index correction, +1′ 28″; height of eye, 15 ft.; time by chronometer, 21 d. 21 h. 6 m. 10 s., which was slow 12.6 s. for mean noon at Greenwich, February 24, and on April 1, was 2 m. 45 s. fast for mean noon at Greenwich. Required the longitude.

8. 1882, August 24, A.M., at ship, latitude at noon, 37° 59′ N.; observed altitude $\underline{\odot}$, 37° 13′ 30″; index correction, +2′ 44″; height of eye, 18 ft.; time by chronometer, August 23 d. 18 h. 13 m. 24 s., which was 1 m. 5 s. fast for mean noon at Greenwich, August 1, and on August 10, was 0 m. 42 s. slow for mean time at Greenwich; course (true) since observation, N.N.W.; distance, 22.4 miles. Required the longitude at noon.

9. 1882, January 29, P.M., at ship, latitude 28° 45′ N.; observed altitude $\underline{\odot}$, 17° 46′ 30″; index correction, −3′ 18″; height of eye, 16 ft.; time by chronometer, January 28 d. 16 h. 31 m. 30 s., which was 1 m. 16.5 s. fast for Greenwich mean noon, December 17, 1881, and on January 1, 1882, was 1 m. 3 s. slow for Greenwich mean time; course (true) since noon, N.W. by W.; distance, 20 miles. Required the longitude at the time of observation, and also at noon.

10. 1882, August 31, P.M., at ship, latitude 0°; observed altitude $\underline{\odot}$, 45° 5′ 30″; index correction, −2′ 4″; height of eye, 15 ft.; time by chronometer, August 31 d. 9 h. 11 m. 28 s., which was 5 m. 20 s. fast for mean noon at Greenwich, April 15, and on June 16, was fast 2 m. 43 s. on mean time at Greenwich. Required the longitude.

11. 1882, April 15, A.M., at ship, latitude 48° 52′ N.; observed altitude $\underline{\odot}$, 22° 18′; index correction, −3′ 54″; height of eye, 17 ft.; time by chronometer, April 14 d. 22 h. 30 m. 42 s., which was 0 m. 4 s. slow for mean noon at Greenwich, January 1, and on January 12, was fast 0 m. 2 s. Required the longitude.

12. 1882, August 28, P.M., at ship, latitude 5° S.; observed altitude $\underline{\odot}$, 38°; index correction, +5′ 27″; height of eye, 21 ft.; time by chronometer, August 27 d. 22 h. 20 m. 30 s., which was 10 m. 0 s. slow for mean noon at Greenwich, February 19, and on May 30, was 2 m. 20 s. slow on mean noon at Greenwich. Required the longitude.

13. 1882, September 22, A.M., at ship, on the equator, observed altitude $\overline{\odot}$, 17° 20′ 40″; index correction, −1′ 9″; height of eye, 20 ft.; time by chronometer, September 22d. 4h. 59m. 16s., which was 15s. slow for Greenwich mean noon, April 30, and on June 1, was 10.6s. fast for mean time at Greenwich. Required the longitude.

14. 1882, August 5, A.M., at ship, latitude at noon 30° 30′ N.; observed altitude $\underline{\odot}$, 35° 6′; height of eye, 15 ft.; time by chronometer, 5d. 8h. 39m. 22s., which was fast 29m. 32.4s. on Greenwich mean noon, July 8, and on July 20, was fast 30m. 0s. on Greenwich mean noon; course (true) till noon, W.; distance, 48 miles. Required the longitude in at noon.

15. 1882, November 12, A.M., at sea, on board the *Swatara*, in latitude 7° 10′ N.; four observed altitudes of the $\underline{\odot}$ were taken at the times (by watch) standing opposite, viz.:

		h.	m.	s.
Obs. alt. $\underline{\odot}$,	21° 8′ 40″	2	55	48
	11′ 50″		56	0
	14′ 50″		56	13
	17′ 30″		56	26.5

Index correction, +31″; height of eye, 18 ft.; correction of watch by chronometer, −5h. 12m. 2.1s. Required the longitude.

16. 1882, November 13, A.M., at sea, on board the *Swatara*, in latitude 9° 30′ N.; five observed altitudes of the $\underline{\odot}$ were taken at the times (by watch) standing opposite, viz.:

		h.	m.	s.
Obs. alt. $\underline{\odot}$,	18° 58′ 40″	2	59	2
	19° 1′ 20″			13
	3′ 30″			28
	7′ 30″			45
	11′ 0″			57.5

Index correction, +32″; height of eye, 18 ft.; correction of watch by chronometer, −5h. 11m. 58.4s. Required the longitude.

17. 1882, November 17, A.M., at sea, on board the *Swatara*, in latitude 15° 35′ N.; five observed altitudes of the ☉ were taken at the times (by watch) standing opposite, viz.:

		h.	m.	s.
Obs. alt. ☉,	23° 56′ 0″	4	12	31
	24° 0′ 0″			46
	4′ 0″		13	2.5
	6′ 10″			14
	10′ 0″			28.5

Index correction, +31″; height of eye, 18 ft.; correction of watch by chronometer, −5h. 11m. 43.6s. Required the longitude.

18. 1882, November 18, A.M., at sea, on board the *Swatara*, in latitude 16° 25′ N.; five observed altitudes of the ☉ were taken at the times (by watch) standing opposite, viz.:

		h.	m.	s.
Obs. alt. ☉,	18° 13′ 30″	3	52	42
	16′ 10″			53.5
	19′ 20″		53	6.5
	22′ 30″			23
	25′ 30″			38

Index correction, +32″; height of eye, 18 ft.; correction of watch by chronometer, −5h. 11m. 39.9s. Required the longitude.

19. 1882, December 4, A.M., at sea, on board the *Swatara*, in latitude 36° 10′ N.; five observed altitudes of the ☉ were taken at the times (by watch) standing opposite, viz.:

		h.	m.	s.
Obs. alt. ☉,	13° 0′ 30″	6	27	14
	3′ 10″		27	29.5
	5′ 40″		27	49
	8′ 50″		28	5
	12′ 0″		28	23

Index correction, +32″; height of eye, 18 ft.; correction of watch by chronometer, −5h. 10m. 47.1s. Required the longitude.

20. 1882, December 4, P.M., at sea, on board the *Swatara*, in latitude 36° 36′ N.; four observed altitudes of the $\underline{\odot}$ were taken at the times (by watch) standing opposite, viz.:

		h.	m.	s.
Obs. alt. $\underline{\odot}$,	16° 31′ 10″	1	7	26.5
	30′ 0″			35.5
	28′ 30″			46
	27′ 20″			56.5

Index correction, +30″; height of eye, 18 ft.; correction of watch by chronometer, −5 h. 10 m. 46.1 s. Required the longitude.

§ 32. Deviation by Time Azimuths.

The **Azimuth** of a heavenly body is the arc of the horizon included between the north or south point and the vertical circle passing through the body. It is named by prefixing the letter (N. or S.) which indicates the point from which it is reckoned, and affixing the letter (E. or W.) which indicates upon which side of the meridian of the observer the body is situated.

The *True Azimuth* is reckoned from the true north or south point.

The *Compass Azimuth* is reckoned from the magnetic north or south point.

When a heavenly body is on the horizon, the arc of the horizon between the body and the east or west point is called the *Amplitude* of the body.

To find the compass azimuth of a heavenly body:

The compass azimuth is observed directly by means of the **Azimuth Compass.**

This instrument differs from the ordinary compass chiefly in having two sight vanes hinged to opposite points of the bowl.

The standard compass is commonly also an azimuth compass.

To observe the compass azimuth, the observer turns the compass so that, by looking through the narrow slit in one of the sight vanes, he can see the object, or the image of it (reflected by a small mirror), bisected by the vertical wire of the other vane. A spring is then touched which holds the card in position until the observer reads off the division of the card, apparently in the prolongation of the wire of the vane.

To find the true azimuth of a heavenly body:

First Method; by computation. The necessary data for time azimuths are: the declination (d), the hour angle (t), and the latitude (L). Let a be the true azimuth.

The first and third formulas of Spher. Trig. § 68 apply.

$$\tan m = \cot d \cos t.$$
$$\tan a \ = \sec(L+m) \tan t \sin m.$$

NOTE. In these formulas m represents the polar distance of the foot of the perpendicular let fall from the body upon the meridian.

Second Method; by inspection. Time azimuth tables have been prepared which give the true azimuth of the sun (and other heavenly bodies whose declinations lie between 23° N. and 23° S.) corresponding to the given latitude, declination, and apparent local time. *This is the best practical method of finding true azimuths.*

To find the total error of the compass:

The total error of the compass is equal to the difference between the true and compass azimuths of a heavenly body, when they are both reckoned from the points of like name toward the east or toward the west; it is equal to the sum of the true and compass azimuths when one is reckoned toward the east and the other toward the west.

To name the total error of the compass, suppose the observer to look from the centre of the compass toward the two azimuths; then the error is east or west according as the true azimuth falls to the right or left of the compass azimuth.

To find the deviation:

If the total error of the compass and the variation (given by the chart) be of the same name (E. or W.), their difference will be the deviation; if of different names, their sum will be the deviation.

When the error exceeds the variation, the deviation is of the same name as the error; but when the error is less than the variation, the deviation is of contrary name to the error.

The deviation is thus determined for the point of the compass toward which the ship's head lies at the moment of observation. By swinging the ship and repeating the observations, the deviation may be found for as many points of the compass as may be desired. Simple interpolation may be employed in finding the deviation for a point situated between two points for which the deviations are known.

ANSWERS.

Exercise I.

1. S. 35° 56′ E.
2. N. 77° 45′ W.
3. S.E. by E.
4. S. 79° 4′ W.
5. N. 82° 47′ W.
6. S. 60° 30′ E.
7. N. 56° 22′ E.
8. S. 73° 26′ W.
9. S. 39° 15′ E.
10. N.E. by E.
11. N.N.W.
12. W.
13. S. 42° 0′ E.
14. N. 30° 0′ W.
15. N. 52° 0′ W.
16. S. 27° 0′ W.
17. S. 17° 0′ W.
18. N. 17° 56′ E.

Exercise II.

1. $L'' =$ 45° 26′ N., $p =$ 405.8 W.
2. $L'' =$ 2° 46′ S., $p =$ 405.6 E.
3. $L'' =$ 0° 3′ S., $p =$ 406.0 E.
4. $L'' =$ 2° 12′ N., $p =$ 52.1 E.
5. $L'' =$ 36° 57′ N., $p =$ 113.1 W.
6. $C =$ 61° 49′, $p =$ 317.3.
7. $C =$ 25° 19′, $p =$ 151.4.
8. $D =$ 118° 3′, $p =$ 23.1 W.
9. $D =$ 199° 6′, $p =$ 58.0 W.
10. $L'' =$ 37° 28′ N., $D =$ 58.0.
11. $L'' =$ 17° 54′ N., $D =$ 72.0.
12. $C =$ 25° 27′, $p =$ 128.5.
13. $C =$ 22° 21′, $p =$ 308.8.
14. $L'' =$ 6° 3′ N., $p =$ 95.4 E.
15. $L'' =$ 14° 3′ S., $p =$ 137.5 E.
16. $L'' =$ 40° 12′ S., $p =$ 239.7 E.
17. $C =$ S. 53° 22′ W., $D =$ 360.4.
18. $C =$ N. 56° 6′ E., $D =$ 154.2.
19. $C =$ N. 75° 5′ E., $D =$ 147.7.
20. $C =$ N. 34° 20′ W., $D =$ 93.3.

EXERCISE III.

1. 505.5 E.
2. 125.0 E.
3. 109.1 W.
4. 246.0 E.
5. 80.1 W.
6. 166° 50′ E.
7. 30° 13′ W.
8. 179° 48′ E.
9. 172° 15′ E.
10. 12° 47′ W.
11. 179° 11′ E.
12. 21° 18′ W.; N. 58° 40′ E.

EXERCISE IV.

1. C = E.N.E., D = 295.3.
2. C = N. 61° 40′ W., D = 210.7.
3. C = S. 64° 34′ W., D = 209.6.
4. L'' = 49° 10′ S., λ'' = 176° 0′ W.
5. L'' = 18° 52′ N., λ'' = 175° 12′ E.
6. L'' = 0° 59′ S., λ'' = 27° 47′ W.
7. L'' = 42° 11′ N., λ'' = 65° 46′ W.
8. L'' = 41° 56′ N., λ'' = 62° 59′ W.
9. L'' = 41° 36′ N., λ'' = 59° 12′ W.
10. λ'' = 57° 55′ W., D = 61.5.
11. λ_d = 3° 42′, C = 87° 15′.
12. C = S. 57° 19′ E., D = 131.5.
13. C = S. 71° 9′ E., D = 510.7.
14. C = S. 53° 46′ E., D = 289.3.
15. C = S. 36° 25′ W., D = 156.6.
16. L'' = 52° 10′ N., λ'' = 69° 13′ W.
17. L'' = 35° 14′ N., λ'' = 73° 2′ W.
18. L'' = 42° 24′ N., λ'' = 66° 15′ W.
19. L'' = 39° 59′ N., λ'' = 68° 35′ W.
20. L'' = 53° 20′ N., λ'' = 1° 25′ W.
21. L'' = 30° 4′ N., λ'' = 67° 27′ W.
22. L'' = 33° 8′ N., λ'' = 66° 19′ W.
23. L'' = N. 64° 55′ W., D = 675.5.
24. L'' = 25° 42′ N., C = S. 36° 27′ W.
    D = 775.8.

EXERCISE V.

1. C = N. $46^\circ\ 29'$ E., $D = 140.9$.
2. C = S. $30^\circ\ 23'$ E., $D = 98.5$.
3. C = S. $28^\circ\ 24'$ E., $D = 261.5$.
4. C = N. $55^\circ\ 13'$ E., $D = 2160$.
5. C = S. $43^\circ\ 39'$ W., $D = 248.8$.
6. $L'' = 47^\circ\ 10'$ N., $\lambda'' = 32^\circ\ 15'$ W.
7. $L'' = 59^\circ\ 3'$ N., $\lambda'' = 2^\circ\ 43'$ E.
8. $L'' = 47^\circ\ 20'$ N., $\lambda'' = 13^\circ\ 45'$ W.
9. $C = 33^\circ\ 20'$, $\lambda_d = 13^\circ\ 10'$.
10. $\lambda'' = 23^\circ\ 5'$ W., $D = 1022$.
11. C = S. $51^\circ\ 59'$ W., $D = 358.8$.
12. C = N. $16^\circ\ 2'$ E., $D = 5599$.
13. C = S. $60^\circ\ 29'$ E., $D = 7685$.
14. $\lambda_d = 4^\circ\ 48'$, $D = 476.3$.
15. By Mercator's, $L'' = 37^\circ\ 16'$ N., $\lambda'' = 22^\circ\ 47'$ W.;
Mid. Lat., $L'' = 37^\circ\ 16'$ N., $\lambda'' = 22^\circ\ 43'$ W.

EXERCISE VI.

1. $L_d = 0^\circ\ 57'$ S., $p = 21.4$ W.
2. $L_d = 1^\circ\ 37'$ S., $p = 45.6$ W.
3. $L_d = 1^\circ\ 39'$ S., $p = 44.6$ W.
4. $L_d = 0^\circ\ 45'$ N., $p = 37.1$ E.
5. $L_d = 0^\circ\ 4'$ S., $p = 25.0$ W.
6. $L_d = 0^\circ\ 15'$ N., $p = 36.5$ W.
7. $L_d = 0^\circ\ 7'$ S., $p = 6.8$ W.
8. $L_d = 1^\circ\ 11'$ S., $p = 110.7$ E.
9. $L_d = 1^\circ\ 2'$ S., $p = 119.5$ E.

EXERCISE VII.

1. $L'' = 45^\circ\ 17'$ N., $\lambda'' = 19^\circ\ 39'$ W.
2. $L'' = 30^\circ\ 49'$ N., $\lambda'' = 17^\circ\ 56'$ W.
3. $L'' = 17^\circ\ 24'$ S., $\lambda'' = 1^\circ\ 28'$ W.
4. $L'' = 46^\circ\ 51'$ N., $\lambda'' = 48^\circ\ 5'$ W.
5. $L'' = 61^\circ\ 34'$ N., $\lambda'' = 149^\circ\ 49'$ E.
6. $L'' = 51^\circ\ 29'$ S., $\lambda'' = 176^\circ\ 35'$ E.
7. $L'' = 36^\circ\ 28'$ N., $\lambda'' = 2^\circ\ 10'$ W.

EXERCISE VIII.

1. Course from the Bermudas = N. 49° 32′ E.;
   Course from the Lizard = N. 88° 42′ W.;
   Distance = 2812 miles; L of V = 49° 59′ N.;
   λ of V = 6° 54′ W.
2. Course from Boston = N. 55° 24′ E.;
   Course from Cape Clear = N. 77° 42′ W.;
   Distance = 2500 miles; L of V = 52° 28′ N.;
   λ of V = 25° 2′ W.
3. Course from Honolulu = N. 32° 3′ E.;
   Course from Vancouver = S. 50° 17′ W.;
   Distance = 2228 miles; L of V = 60° 22′ N.;
   λ of V = 80° 41′ W.
4. Course from Sandy Hook = N. 52° 50′ E.;
   Course from Cape Clear = N. 78° 18′ W.;
   Distance = 2698 miles; L of V = 52° 23′ N.;
   λ of V = 33° 42′ W.
5. Course from Cape Frio = N. 23° 2′ E.;
   Course from the Lizard = S. 34° 4′ W.;
   Distance = 4806 miles; L of V = 68° 51′ N.;
   λ of V = 34° 17′ E.
6. Course from Cape Good Hope = S. 85° 13′ W.;
   Course from Cape Frio = S. 63° 23′ E.;
   Distance = 3208 miles; L of V = 34° 37′ S.;
   λ of V = 10° 3′ E.
7. Course from Perth = S. 87° 45′ E.;
   Course from Grand Port = S. 64° 37′ W.;
   Distance = 3158 miles; L of V = 32° 8′ S.;
   λ of V = 111° 29′ E.
8. Course from A = N. 43° 57′ E.;
   Course from B = S. 83° 51′ W.;
   Distance = 3714 miles; L of V = 48° 19′ N.;
   λ of V = 3° 39′ E.

9. By Great Circle:
Course from A = S. 49° 28′ E.;
Course from B = S. 36° 32′ W.;
Distance = 6772 miles; L of V = 54° 24′ S.;
λ of V = 95° 26′ E.

By Rhumb-line:
Course from A = N. 76° 58′ E.;
Course from B = S. 76° 58′ W.;
Distance = 7432 miles.

EXERCISE IX.

1. 25° 1′ 20″.
2. 15° 10′ 41″.
3. 18° 27′ 11″.
4. 30° 23′ 4″.
5. 56° 35′ 28″.
6. 60° 19′ 5″.
7. 31° 35′ 13″.
8. 25° 57′ 27″.
9. 20° 13′ 41″.
10. 35° 51′ 13″.

EXERCISE X.

			d.	h.	m.	s.					d.	h.	m.	s.
1.	1882	July	8	7	6	10 P.M.	6.	1882	June	30	23	8	25.
2.	1882	Mar.	8	0	25	30 A.M.	7.	1880	Mar.	2	11	56	56.
3.	1880	Jan.	2	6	10	10 A.M.	8.	1880	Aug.	31	10	8	20.
4.	1880	Jan.	1	3	0	0 A.M.	9.	1881	Aug.	31	12	12	15.
5.	1883	Feb.	2	8	4	30 P.M.	10.	1882	Dec.	31	22	41	56.

EXERCISE XI.

		d.	h.	m.	s.			d.	h.	m.	s.
1.	May	4	17	35	35.	6.	Dec.	31	6	7	0 A.M.
2.	July	31	1	53	50.	7.	July	4	11	55	0 P.M.
3.	July	31	19	32	58.	8.	July	4	11	55	0 P.M.
4.	Mar.	2	6	57	0.	9.	May	20	7	43	20 A.M.
5.	Mar.	25	17	49	42.	10.	1881. Jan.	1	3	16	40 A.M.

EXERCISE XII.

		m.	s.
1.	22° 20′ 7.4″ S.;	6	38.85 to be added to app. time.
2.	17° 55′ 52.1″ N.;	6	3.66 to be added to app. time.
3.	16° 25′ 30.3″ N.;	3	30.82 to be subtracted from a.t.

m. s.
4. 16° 13′ 4.7″ N.; 5 21.87 to be added to app. time.
5. 22° 18′ 47.4″ S.; 9 28.67 to be subtracted from a.t.
6. 19° 53′ 5.6″ N.; 6 14.27 to be added to app. time.
7. 14° 32′ 17.6″ S.; 16 18.75 to be subtracted from a.t.
8. 7° 32′ 47.4″ S.; 13 31.88 to be subtracted from a.t.
9. 22° 47′ 20.4″ N.; 1 25.88 to be added to mean time.
10. 16° 20′ 41.1″; 14 7.32 to be subtracted from m.t.

Exercise XIII.

1. 6° 49′ 9″ N.
2. 5° 23′ 43″ S.
3. 0.
4. 45° 47′ 9″ N.
5. 30° 30′ 40″ N.
6. 43° 43′ 32″ S.
7. 16° 0′ 55″ N.
8. 0° 13′ 46″ N.
9. 4° 32′ 48″ N.
10. 49° 16′ 24″ N.
11. 5° 43′ 3″ S.
12. 6° 7′ 30″ N.
13. 29° 26′ 44″ S.
14. 0° 24′ 44″ S.
15. 50° 22′ 22″ S.
16. 7° 52′ 4″ N.
17. 6° 31′ 49″ S.

Exercise XIV.

1. 53° 45′ 44″ N.
2. 18° 17′ 0″ N.
3. 34° 5′ 7″ S.
4. 14° 38′ 0″ S.
5. 60° 38′ 37″ S.
6. 16° 19′ 2″ N.
7. 50° 28′ 45″ N.
8. 29° 30′ 21″ S.
9. 52° 31′ 41″ N.
10. 35° 40′ 35″ N.
11. 45° 34′ 15″ S.
12. 48° 24′ 1″ S.
13. 46° 4′ 33″ S.

Exercise XV.

1. 16° 51′ 30″ E.
2. 13° 50′ 45″ E.
3. 12° 57′ 30″ E.
4. 10° 52′ 30″ E.
5. 49° 20′ 15″ W.
6. 41° 31′ 0″ W.
7. 20° 0′ 45″ W.
8. 35° 27′ 15″ E. at sights.
35° 16′ E. at noon.
9. 170° 54′ 30″ E. at sights.
171° 13′ E. at noon.
10. 93° 51′ 0″ W.
11. 44° 30′ 30″ W.
12. 76° 4′ 0″ E.
13. 149° 18′ 30″ W.
14. 179° 17′ 30″ W. at sights.
179° 47′ E. at noon.
15. 34° 43′ 30″ W.
16. 36° 50′ 15″ W.
17. 45° 51′ 0″ W.
18. 47° 5′ 0″ W.
19. 74° 19′ 15″ W.
20. 75° 16′ 30″ W.